Large scale behaviour of 3D continuous phase coexistence models

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Abstract

We study a class of three dimensional continuous phase coexistence models, and show that, under different symmetry assumptions on the potential, the large-scale behaviour of such models near a bifurcation point is described by the dynamical Φ^p_3 models for $p \in \{2, 3, 4\}$. This result is specific to space dimension 3 and does not hold in dimension 2.

Contents

1 Introduction

The aim of this article is to study the large scale behavior of phase coexistence models of the type

$$\partial_t u = \Delta u - \epsilon V_\theta'(u) + \delta\widehat{\xi}\;, \tag{1.1}$$

in three spatial dimensions, where V_θ denotes a potential depending on some parameter θ and ϵ, δ are two small parameters. Throughout this article, $\widehat{\xi}$ is assumed to be a continuous space-time Gaussian random field modelling the local fluctuations, with covariance having compact support and integrating to 1. The potential $(\theta, u) \mapsto V_\theta(u)$ is a sufficiently regular function (depending on the regime, we will actually assume that it is polynomial in u). Regarding the two parameters ϵ and δ, we will consider two extremal regimes: either $\epsilon = o(1), \delta \approx 1$, which we call the weakly nonlinear regime, or $\delta = o(1), \epsilon \approx 1$, which we call the weak noise regime. However, our results would easily carry over to intermediate regimes as well. Also, the spatial domain of the process u is a large three dimensional torus whose size depends on ϵ (see Remark 1.2 for more details).

For the sake of the present discussion, consider the weakly nonlinear regime, i.e. set $\delta = 1$ in (1.1). It is then natural to consider scalings of the type $u_\lambda(t,x) = \lambda^{-1/2}u(t\lambda^{-2}, x\lambda^{-1})$ which leave invariant the stochastic heat equation, so that u_{ϵ^α} satisfies

$$\partial_t u_{\epsilon^\alpha} = \Delta u_{\epsilon^\alpha} - \epsilon^{1-5\alpha/2} V_\theta'(\epsilon^{\alpha/2} u_{\epsilon^\alpha}) + \xi_{\epsilon^\alpha} , \tag{1.2}$$

where $\xi_{\bar\epsilon}$ denotes a suitable rescaling of $\widehat{\xi}$ which approximates space-time white noise at scales larger than $\bar\epsilon$.

Remark 1.1. Since the process u in (1.1) itself depends on ϵ, one should really write $u_{\epsilon,\epsilon^\alpha}$ for the rescaled process in (1.2) to avoid ambiguity. However, we still write the ambiguous one u_{ϵ^α} here in order to keep the notations simple.

The form (1.2) suggests that if we start (1.1) with an initial condition located at a local minimum of V, then at scales of order $\epsilon^{-1/2}$ (i.e. setting $\alpha = \frac{1}{2}$ in (1.2)) solutions should be well approximated by solutions to an Ornstein-Uhlenbeck process of the type

$$\partial_t v = \Delta v - cv + \xi , \tag{1.3}$$

for some $c > 0$ and ξ a space-time white noise. As we will see in Theorem 5.2 below, this is in general false, unless V is harmonic to start with. Instead, one should compute from V_θ an effective potential $\langle V_\theta \rangle$ in the following way. Consider the space-time stationary solution Ψ to the linearised equation

$$\partial_t \Psi = \Delta \Psi + \widehat{\xi} . \tag{1.4}$$

Since we are in dimension 3, such a solution exists and is Gaussian with finite variance C_0. We then set

$$\langle V_\theta \rangle(x) = \int_{\mathbf{R}} V_\theta(x+y)\mu(dy),$$

where $\mu = \mathcal{N}(0, C_0)$. In other words, $\langle V_\theta \rangle$ is the effective potential obtained by averaging V against the stationary measure of Ψ. We show in Theorem 5.2 that if we start with an initial condition located at a local minimum of $\langle V_\theta \rangle$, then it is indeed the case that the behaviour at scales of order $\epsilon^{-1/2}$ is described by (1.3).

These considerations suggest that more interesting nonlinear scaling limits can arise in regimes where $\theta \mapsto \langle V_\theta \rangle$ undergoes a bifurcation, and this is the main object of study of this article. In particular, if $\langle V_\theta \rangle$ is symmetric and undergoes a pitchfork bifurcation at some $\theta = \theta_0$, then one would expect the large-scale behaviour to be described near θ_0 by the dynamical Φ^4_3 model built in [Hai14b] and further investigated in [CC13, Kup15]. Similarly, near a saddle-node bifurcation, one would expect the large-scale behaviour to be described by the dynamical Φ^3_3 model built in [EJS13] using the techniques developed in [DPD02, DPD03].

Recall that, at least formally, the dynamical Φ^p_3 model is given by

$$\partial_t \Phi = \Delta \Phi - \Phi^{p-1} + \xi, \tag{1.5}$$

where ξ is the space-time white noise, and the spatial variable belongs to the three-dimensional torus $\mathbf{T}^3$. In this article, we will only ever consider $p \in \{2, 3, 4\}$, with $p = 2$ corresponding to the Ornstein-Uhlenbeck process (1.3). For $p \in \{3, 4\}$, the interpretation of (1.5) is not clear a priori since solutions are distribution-valued so that the term Φ^{p-1} lacks a canonical interpretation. However, they can be constructed as limits of solutions to

$$\partial_t \Phi_\epsilon = \Delta \Phi_\epsilon - \Phi_\epsilon^{p-1} + C_\epsilon \Phi_\epsilon^{p-3} + \xi_\epsilon \;, \tag{1.6}$$

for a regularisation ξ_ϵ of space-time white noise and a suitable diverging sequence of constants C_ϵ. In the case $p = 3$, this turns the term Φ^2 into the Wick product $:\!\Phi^2\!:$ with respect to the Gaussian structure induced by the stationary solution to the corresponding linearised equation (see [EJS13] for more details). In the case $p = 4$, the situation is more delicate and additional logarithmic divergencies arise due to higher order effects, see [GJ73, Fel74, Hai14b].

At this stage, it is important to note that the notation (1.5), even when interpreted as limit of processes of the type (1.6), is really an abuse of notation: since one could always change the value of C_ϵ in (1.6) by a finite quantity, one actually has a one-parameter family of solutions indexed by that finite quantity, and we call the resulting family of solutions Φ_3^p family. Let us point out that, without the presence of the diverging counterterm C_ϵ, the sequence Φ_ϵ for $p = 4$ would converge to 0 in a sufficiently weak topology depending on the dimension d (see [HRW12] for more details).

Formally, the equilibrium measure of the dynamics (1.5) for $p = 4$ is the measure on Schwartz distributions associated to Bosonic Euclidean quantum field theory. This can also be justified rigorously, see [HM15]. The construction of this measure was a major achievement of constructive field theory; see the articles [EO71, Fel74, FO76, GJ73, Gli68] and references therein. In two spatial dimensions, the equation (1.5) was treated in [AR91, DPD03, MW15]. For $d \geq 4$, one does not expect to be able to obtain any non-trivial scaling limit, see [Frö82, Aiz82, BBS14].

Another reason why the dynamical Φ_3^4 is interesting is that it is expected to describe the 3D Ising model with Glauber dynamics and Kac interactions near critical temperature (as conjectured in [GLP99]). In fact, the one dimensional version of this result was shown in [BPRS93] at the critical temperature. The two dimensional case is more difficult, as the equation itself requires renormalisation. It was shown recently in [MW14] that the 2D Kac-Ising model does rescale to Φ_2^4 near critical temperature, and the renormalisation constant has a nice interpretation as the shift of critical temperature from its mean field value. See also the article [GS73] which however required a two-step procedure to obtain Φ_2^4 from an Ising model.

We now turn back to the rescaled process (1.2). As suggested by the form of renormalisation in (1.6), it is reasonable to expect that the behaviour of u_ϵ at scale $\alpha = 1$ and θ at (or near) a pitchfork bifurcation should be well approximated by the dynamical Φ_3^4 model. However, it turns out that this is *not* true in full generality. The main result of this article is that, although u_ϵ converges to Φ_3^4 for all symmetric polynomial potentials, for generic asymmetric potentials, after proper re-centering and rescaling, the large scale behaviour of the system will always be described either by Φ_3^3 or by the O.U. process of the type (1.3). One way to understand this is that, as is well-known from dynamical systems, pitchfork bifurcations are structurally unstable: small generic perturbations tend to turn them into a saddle-node bifurcation taking place very close to a local minimum. One can then argue (this is quite clear in Wilson's renormalisation group picture which has recently been applied to the construction of the dynamical Φ^4_3 model in [Kup15]) that the effective potential experienced by the process at large scale is not $\langle V_\theta \rangle$ but some small perturbation thereof, thus reconciling our results with intuition.

1.1 Weakly nonlinear regime

We start with the weakly nonlinear regime given by

$$\partial_t u = \Delta u - \epsilon V'_\theta(u) + \widehat{\xi}, \tag{1.7}$$

where we assume that V_θ is a *polynomial* whose coefficients depend smoothly on θ. Defining $\langle V_\theta \rangle$ as above, we thus write

$$\langle V'_\theta \rangle(u) = \sum_{j=0}^{m} \widehat{a}_j(\theta) u^j \ ,$$

for some smooth functions $\widehat{a}_j$. For notational simplicity, we let $\widehat{a}_j, \widehat{a}'_j$ and $\widehat{a}''_j$ denote the value and first two derivatives of $\widehat{a}_j(\theta)$ at 0. We will always assume that $\langle V_\theta \rangle$ has a critical point at the origin (which could easily be enforced by just translating u), so that $\widehat{a}_0 = 0$.

Remark 1.2. From now on, we will always assume that (1.7) is considered on a periodic domain of the relevant size. In particular, we define u_{ϵ^α} directly as the solution to (1.2) on a domain of size $\mathcal{O}(1)$ (the precise size is irrelevant, but it should be bounded and no longer depend on ϵ). Ideally, one would like to extend the convergence results of this article to all of $\mathbf{R}^3$, which would be much more canonical, but this requires some control at infinity which is lacking at present.

Remark 1.3. In principle, the noise $\widehat{\xi}$ appearing in (1.7) also depends on ϵ, since it is defined on a torus of size $\epsilon^{-\alpha}$ for some $\alpha > 0$ depending on the regime we consider. However, since we assume that its correlation function is fixed (independent of ϵ) and has compact support, the noises on domains of different sizes agree in law when considered on an identical patch, as long as a suitable fattening of that patch remains simply connected.

In the simplest case when $\widehat{a}_1 \neq 0$, it is not very difficult to show that at scale $\alpha = \frac{1}{2}$, u_{ϵ^α} converges in probability to the O.U. process. Interesting phenomena occur when $(0,0)$ is a bifurcation point for $\langle V_\theta \rangle$, which gives the necessary bifurcation condition

$$\widehat{a}_0 = \widehat{a}_1 = 0 \ . \tag{1.8}$$

The saddle-node bifurcation further requires that $\widehat{a}'_0 \neq 0$ and $\widehat{a}_2 \neq 0$, and in this case one should choose $\alpha = \frac{2}{3}$ so that as long as $\theta = \mathcal{O}(\epsilon^{\frac{2}{3}})$, the macroscopic process u_{ϵ^α} converges to Φ^3_3 family. In fact, the terms in $V'_\theta(\epsilon^{\alpha/2} u_{\epsilon^\alpha})$ in (1.2) are Hermite polynomials in u_{ϵ^α} whose coefficients are precisely $\widehat{a}_j(\theta)$'s with corresponding powers of ϵ. Thus, the Wick renormalisation is already taken account of, and this is the reason why the bifurcation assumption naturally appears for $\langle V_\theta \rangle$ but not V_θ.

The most interesting case arises when $(0, 0)$ is a pitchfork bifurcation point of $\langle V_\theta \rangle$ so that in addition to (1.8), one has

$$\widehat{a}_0' = 0, \quad \widehat{a}_1' < 0, \quad \widehat{a}_2 = 0, \quad \widehat{a}_3 > 0 \; . \tag{1.9}$$

As mentioned above, from (1.6), it is natural to expect that at scale $\alpha = 1$, and with a suitable choice of θ, the processes u_{ϵ^α} should converge to the solution of the Φ^4_3 model. As already alluded to earlier, this turn out to be true if and only if the quantity

$$A = \int P(z) \;\; \mathbf{E}(V_0'(\Psi(0))V_0''(\Psi(z)))dz \tag{1.10}$$

vanishes, where P is the heat kernel, z denotes the space time variable (t, x), and the expectation is taken with respect to the stationary measure of Ψ as defined in (1.4). For general V_0, this integral diverges since the heat kernel P is not integrable at large scales. It turns out however that this expression is finite provided that

$$\widehat{a}_0 \widehat{a}_1 = \widehat{a}_2 = 0 \; ,$$

which is certainly the case when $\langle V_\theta \rangle$ has a pitchfork bifurcation at the origin. The quantity A can be written in terms of the coefficients of $\langle V \rangle$ as

$$A = \sum_{j=3}^{m-1} (j+1)! \cdot \widehat{a}_j \widehat{a}_{j+1} C_j, \tag{1.11}$$

where the C_j (to be defined in Section 4 below) are explicit constants depending only on the covariance of $\widehat{\xi}$. It is clear from this expression that A vanishes if V is symmetric.

If $A \neq 0$ then, in order to obtain a nontrivial limit, it is necessary to slightly shift the potential from the origin, so we set

$$u_{\epsilon^\alpha}(t, x) = \epsilon^{-\frac{\alpha}{2}} (u(t/\epsilon^{2\alpha}, x/\epsilon^\alpha) - h_\epsilon) \; , \tag{1.12}$$

for some small h_ϵ. The process u_{ϵ^α} above then satisfies the equation

$$\partial_t u_{\epsilon^\alpha} = \Delta u_{\epsilon^\alpha} - \epsilon^{1-5\alpha/2} V_\theta'(\epsilon^{\alpha/2} u_{\epsilon^\alpha} + h_\epsilon) + \xi_{\epsilon^\alpha}. \tag{1.13}$$

From now on, in both weakly nonlinear and weak noise regimes, we will use u_{ϵ^α} to denote the re-centred process, and the process in (1.2) is a special case of (1.13) when $h = 0$.

If one then takes $\theta \sim \epsilon^\beta$ for some $\beta < \frac{2}{3}$, then there are three different choices of h_ϵ's such that the shifted process u_{ϵ^α} converges to O.U. for $\alpha = \frac{1+\beta}{2}$. As expected, two of the possible limiting O.U. processes are stable, and the third one is unstable. If $\theta \sim \epsilon^\beta$ for some $\beta > \frac{2}{3}$ on the other hand, then there is a unique choice of h_ϵ such that at scale $\alpha = \frac{5}{6}$, the process u_{ϵ^α} converges to a stable O.U. process.

At the critical case $\theta = c\epsilon^{\frac{2}{3}}$, there is a constant c^* such that for $c > c^*$ and $c < c^*$, at scale $\alpha = \frac{5}{6}$, u_{ϵ^α} either converges to three O.U.'s or just one O.U., respectively. At

$c = c^*$, there are two possible choices of h_ϵ. One of them again yields a stable O.U. process at scale $\frac{5}{6}$ in the limit, but the other one yields Φ^3_3 at scale $\alpha = \frac{8}{9}$. Note that this scale is much larger than the scale $\frac{2}{3}$ at which one obtains Φ^3_3 in the case of a simple saddle-node bifurcation. We summarise them in the following theorem.

Theorem 1.4. *Let $\langle V_\theta \rangle$ have a pitchfork bifurcation at the origin, and let u_{ϵ^α} be the solution to* (1.2) *on* $[0, T] \times \mathbf{T}^3$.

If the quantity A given by (1.10) *is* 0*, then there exists $\mu < 0$ such that at the distance to criticality*

$$\theta = \mu\epsilon|\log \epsilon| + \lambda\epsilon + \mathcal{O}(\epsilon^2),$$

scale $\alpha = 1$ and $h = 0$, the process u_ϵ converges to the Φ^4_3 family indexed by λ.

If $A \neq 0$, then the large scale behaviour of u^α_ϵ depends on the value

$$\theta = \rho\epsilon^\beta, \rho > 0.$$

In fact, there exists $\rho^* > 0$ such that if $\beta < \frac{2}{3}$, or if $\beta = \frac{2}{3}$ and $\rho > \rho^*$, then there exist three choices of h_ϵ's such that at scale $\alpha = \frac{1+\beta}{2}$, two of the resulting processes u_{ϵ^α} converge to a stable OU process, and the other converges to an unstable one.

If $\beta > \frac{2}{3}$, or if $\beta = \frac{2}{3}$ and $\rho < \rho^*$, then there exists a choice of h_ϵ such that at scale $\alpha = \frac{5}{6}$, the process u_{ϵ^α} converges to a stable OU process.

At the critical value $\beta = \frac{2}{3}$ and $\rho = \rho^*$, there exist two choices of h_ϵ such that one of the resulting processes converges to a stable OU process at scale $\alpha = \frac{5}{6}$, and the other converges to Φ^3_3 at scale $\alpha = \frac{8}{9}$.

Remark 1.5. The coefficient of the Wick term $:u^2:$ in the critical Φ^3_3 case is proportional to $A^{\frac{1}{3}}$. If $A = 0$, then the process becomes a free field, and one can then further enlarge the scale to 1, and adjust θ and h to get Φ^4_3.

Also, the coefficient of the term Φ^{p-1} in the limiting equation depends on various coefficients of $\langle V_0 \rangle$, but we can rescale them while leaving invariant the white noise such that they all become 1.

Remark 1.6. In the asymmetric case ($A \neq 0$), one can actually expand θ to the second order such that in the branch containing the saddle point, the scale increases continuously from 0 up to $\frac{8}{9}$ with respect to θ (see Remark 5.5). Similar results also hold in the symmetric case, but this is not important here, so we omit the details.

All these results will be formulated precisely in Section 5 below.

1.2 Weak noise regime

There is another regime of microscopic models in which the nonlinear dynamics dominates the noise. The local mean field fluctuation is given by the equation

$$\partial_t u = \Delta u - V'_\theta(u) + \epsilon^{\frac{1}{2}}\widehat{\xi}, \tag{1.14}$$

where V_θ is a potential with sufficient regularity, not necessarily a polynomial. More precisely, we assume $V : \theta \mapsto V_\theta(\cdot)$ is a smooth function in the space of $\mathcal{C}^8$ functions.

Thus, we can Taylor expand V'_θ around $x = 0$ as

$$V'_\theta(x) = \sum_{j=0}^{6} a_j(\theta)x^j + F_\theta(x), \tag{1.15}$$

where a_j's are smooth functions in θ, and $|F_\theta(x)| \lesssim |x|^7$ uniformly over $|\theta| < 1$ and $|x| < 1$.

Since the noise now has strength of order $\epsilon^{\frac{1}{2}}$, the large scale behaviour of (1.14) is determined by the behaviour of V_θ itself near the origin, and not by that of an effective potential. Again, in order to observe an interesting limit, we assume that V has a pitchfork bifurcation at $(0, 0)$, namely one has

$$a_0 = a'_0 = a_1 = a_2 = 0, \qquad a'_1 < 0, \qquad a_3 > 0, \tag{1.16}$$

where the $a_j(\theta)$ are the coefficients of the Taylor series of $V'_\theta(\phi)$ around $\phi = 0$. For $\lambda > 0$, we set similarly to before

$$u_\lambda(t, x) = \lambda^{-1}(u(t\lambda^{-2}, x\lambda^{-1}) - h_\epsilon)\;,$$

where h_ϵ is a small parameter as before. By setting $\lambda = \epsilon^\alpha$, we see that this time u_{ϵ^α} solves the PDE

$$\partial_t u_{\epsilon^\alpha} = \Delta u_{\epsilon^\alpha} - \epsilon^{-(\frac{1}{2}+\frac{5\alpha}{2})} V'_\theta(\epsilon^{\frac{1}{2}+\frac{\alpha}{2}} u_{\epsilon^\alpha} + h_\epsilon) + \xi_{\epsilon^\alpha}\;. \tag{1.17}$$

While this appears to be identical to (1.2) modulo the substitution $\alpha \mapsto \alpha + 1$, it genuinely differs from it in that the driving noise still has correlation length ϵ^α and not $\epsilon^{\alpha+1}$. In order for u_{ϵ^α} to converge to Φ^4_3, it then seems natural to choose $\alpha = 1$, thus guaranteeing that the coefficient of the cubic term in the Taylor expansion of V'_θ is of order 1. But this creates the divergences in both linear and constant terms on the right hand side of the equation. Since $a_0 = a_1 = 0$, and we have two parameters θ and h to tune, it looks like that we could kill the divergences by choosing the proper values of θ and h and get Φ^4_3 in the limit.

Unfortunately, this turns out to be impossible. When tuning θ to its correct value to kill the linear divergence, the terms involving the leading order of h will be precisely be canceled out so that h could only have a second order effect, which is far from enough to kill the divergence in the constant term. Thus, one cannot make both linear and constant terms convergent unless the coefficients of V itself are balanced. It turns out that similar to before, whether u_ϵ converges to Φ^4_3 depends on the quantity

$$B = a_4 + \frac{3a''_0 a_3^2}{2a'^2_1} - \frac{a'_2 a_3}{a'_1}\;.$$

The main statement is the following.

Theorem 1.7. *Assume $V : \theta \mapsto V_\theta(\cdot)$ is a smooth function in the space of $\mathcal{C}^8$ functions, and exhibits a pitchfork bifurcation at the origin $(\theta, x) = (0, 0)$. Let u_{ϵ^α} solves the PDE* (1.17).

If $B = 0$, then there exist choices of θ and h of the form

$$\theta = a\epsilon + b\epsilon^2 \log \epsilon + \mathcal{O}(\epsilon^2) \;, \qquad h = \mathcal{O}(\epsilon) \;,$$

such that u_{ϵ^α} converges to $\Phi^4_3(a_3)$ with an additional constant in the equation.

If $B \neq 0$, then there exist $\rho_j^* > 0$ for $j = 1, 2, 3$ such that if

$$\theta = \theta^* = \rho_1^* \epsilon + \rho_2^* \epsilon^{\frac{4}{3}} + \rho_3^* \epsilon^{\frac{5}{3}} + \mathcal{O}(\epsilon^{\frac{16}{9}}),$$

then there exist two choices of h_ϵ such that one of the resulting processes u_{ϵ^α} converges to Φ^3_3 at scale $\alpha = \frac{7}{9}$, and the other one converges to a stable OU process at $\alpha = \frac{2}{3}$.

If $\theta > \theta^*$ (resp. $\theta < \theta^*$), then there exist three (resp. one) choices of h_ϵ such that the resulting u_{ϵ^α} converge to OU processes. In the former case, two of the OU processes are stable and the last one is unstable; in the latter case the OU process is stable.

Remark 1.8. Similar to the weakly nonlinear case, the coefficient of the Wick term for Φ^3_3 is proportional to $B^{\frac{1}{3}}$. Also, one could rescale the solution leaving invariant the white noise such that all the limits are of the form (1.5).

The precise statement will be given in Theorems 5.8 and 5.10.

1.3 Some remarks and structure of the article

Before describing the structure of this article, we discuss two possible natural generalisations of our results.

1. We expect that analogous results still hold when the noise $\widehat{\xi}$ is not assumed to be Gaussian, but still satisfies good enough integrability and mixing conditions. The techniques developed in [HS15] should apply here as well. Note however that if the noise is asymmetric, then we do not expect to see Φ^4_3 at large scales generically, even if V_θ is symmetric.

2. The assumption that V_θ is a polynomial can probably be relaxed. It is however not clear at all at this stage how the methods in this article could be carried over to handle this case.

It turns out that, as in [HQ15], the weak noise regime can be treated as a perturbation of the weakly nonlinear regime, so we will mainly focus on the latter case. The main strategy to prove the above results is the recently developed theory of regularity structures ([Hai14b]), combined with the results of ([HQ15]), where results analogous to ours are obtained for the KPZ equation. The idea is to lift and solve (1.2) in an abstract regularity structure space that is purposed built for this equation, and then pull the solution back to the usual distribution spaces after suitable renormalisation.

The article is organised as follows. In Section 2, we construct the regularity structure as well as the renormalisation maps that allow us to treat the equations of the form (1.2). Section 3 is devoted to construction of the solution to the abstract equation. In Section 4, we prove the convergence of the renormalised models. Finally, in Section 5, we collect all the previous results to identify the limit of the renormalised solutions.

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2 Construction of the regularity structure

In this section, we build a regularity structure that is sufficiently rich to solve the fixed point problem for the equation

$$\partial_t u_\epsilon = \Delta u_\epsilon - \epsilon^{-\frac{3}{2}} V_\theta'(\epsilon^{\frac{1}{2}} u_\epsilon) + \xi_\epsilon \tag{2.1}$$

in the abstract space of modelled distributions. Here, ξ_ϵ is a mollified version of the space-time white noise ξ at scale ϵ, and V_θ' is a polynomial of degree m. Note that (2.1) corresponds to the weakly nonlinear regime with scale $\alpha = 1$, and we do not restrict V to be symmetric here. Since this is the largest scale we will look at, all other situations (including the weak noise regime) will follow as a perturbation of the above equation.

The construction of the regularity structure mainly follows the methodologies and set up in [Hai14b] and [HQ15], with some slight modifications to accommodate the particular form of the equation (2.1). More gentle introductions to regularity structures can be found in [Hai15a], [Hai15b], [Hai14a] and [CW15].

2.1 The (extended) regularity structure

Recall that a regularity structure is a pair $(\mathcal{T}, \mathcal{G})$, where $\mathcal{T} = \bigoplus_{\alpha \in A} \mathcal{T}_\alpha$ is a vector space that is graded by some (bounded below, locally finite) set $A \subset \mathbf{R}$ of homogeneities, and $\mathcal{G}$ is a group of linear transformations of $\mathcal{T}$ such that, for every $\Gamma \in \mathcal{G}$, $\Gamma - \mathrm{id}$ is strictly upper triangular with respect to the graded structure.

For the purpose of this article, we build basis vectors $\mathcal{T}$ similarly to [HQ15] as a collection of formal expressions built from the symbols $\mathbf{1}$, Ξ, $\{X_i\}_{i=0}^3$ and operators $\mathcal{I}$ and $\mathcal{E}^\beta$ for *half integers* $\beta > 0$. As usual, we assume that all symbols and sub-expressions commute and that $\mathbf{1}$ is neutral for the product, so we identify for example $\mathcal{I}(\Xi X_1)\Xi$ and $\Xi \mathbf{1} \mathcal{I}(X_1 \Xi)$. Given a multi-index $k = (k_0, \cdots, k_3)$, we also write X^k as a shorthand for $X_0^{k_0} \cdots X_3^{k_3}$ (with the convention $X_i^0 = \mathbf{1}$), and $|k| = 2k_0 + \sum_{i=1}^3 k_i$ for its parabolic degree.

With these notations, we define two sets $\mathcal{U}$ and $\mathcal{V}$ of such expressions as the smallest sets such that $X^k \in \mathcal{U}$, $\Xi \in \mathcal{V}$, and such that for every $k \in \{1, \dots, m-3\}$,

$$\begin{aligned} \{\tau_1, \cdots, \tau_k\} \subset \mathcal{U} \qquad &\Rightarrow \qquad \{\tau_1\tau_2\tau_3\ ,\ \mathcal{E}^{\frac{k}{2}}(\tau_1 \cdots \tau_{k+3})\} \subset \mathcal{V}\ , \\ \tau \in \mathcal{V} \qquad &\Rightarrow \qquad \mathcal{I}(\tau) \in \mathcal{U}\ . \end{aligned} \tag{2.2}$$

We then set $\mathcal{W} = \mathcal{U} \cup \mathcal{V}$ and we associate to each element of $\mathcal{W}$ a homogeneity in the following way. We set

$$|\Xi| = -\frac{5}{2} - \kappa, \qquad |X^k| = |k|\ ,$$

where κ is a small positive number to be fixed later, and we extend this to every formal expression in $\mathcal{W}$ by

$$|\tau\bar{\tau}| = |\tau| + |\bar{\tau}|, \quad |\mathcal{I}(\tau)| = |\tau| + 2, \quad |\mathcal{E}^\beta(\tau)| = \beta + |\tau|\ . \tag{2.3}$$

We then write $\mathcal{T}_\alpha$ for the free vector space generated by $\{\tau \in \mathcal{W} : |\tau| = \alpha\}$. In this article, we will only ever use basis vectors with homogeneity less than 2, we therefore take for $\mathcal{T}$ the space of all finite linear combinations of elements of $\mathcal{W}$ of homogeneity less than 2, i.e. $\mathcal{T} = \bigoplus_{\alpha<2} \mathcal{T}_\alpha$.

The main reason for introducing $\mathcal{E}^\beta$ as in (2.2) rather than treating ϵ as a fixed real number is the following crucial fact. It reflects that (2.1) is subcritical under the scaling reflected by our regularity structure.

Lemma 2.1. *If $\kappa < \frac{1}{8m}$, then for every $\gamma > 0$, the set $\{\tau \in \mathcal{W} : |\tau| < \gamma\}$ is finite.*

As in [HQ15], it will be convenient to consider $\mathcal{E}^\beta$ as a linear map such that $\mathcal{E}^\beta : \tau \mapsto \mathcal{E}^\beta(\tau)$. The problem is that the product $\tau_1 \cdots \tau_{\ell+3}$ appearing in (2.2) does in general not belong to $\mathcal{T}$. Just as in [HQ15], one way to circumvent this problem is to introduce the extended regularity structure $\mathcal{T}_{\text{ex}}$, given by the linear span of

$$\mathcal{W}_{\text{ex}} = \mathcal{W} \cup \{\tau_1 \cdots \tau_m : \tau_j \in \mathcal{U}\}.$$

In this way, we can view $\mathcal{E}^\beta$ as a linear map defined on (a subspace of) $\mathcal{T}_{\text{ex}}$.

We now start to describe the structure group $\mathcal{G}$ for $\mathcal{T}_{\text{ex}}$. For this, we introduce the following three sets of formal symbols:

$$\begin{aligned} &\mathcal{F}_1 = \{\mathbf{1}, X\}, \qquad \mathcal{F}_2 = \{\mathscr{I}_\ell(\tau) : \ \tau \in \mathcal{W} \setminus \{X^k\}, \ |\tau| + 2 > \ell\}, \\ &\mathcal{F}_3 = \{\mathscr{E}_\ell^{\frac{k}{2}}(\tau_1 \cdots \tau_{k+3}) : \ \tau_j \in \mathcal{U}, \ \frac{k}{2} + \sum_j |\tau_j| > |\ell| \geq \sum_j |\tau_j|\} \ . \end{aligned} \tag{2.4}$$

We then let $\mathcal{T}_+$ be the commutative algebra generated by the elements in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ and we define a linear map $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{T}_+$ in the same way as in [HQ15, Section 3.1].

For any linear functional $g : \mathcal{T}_+ \to \mathbf{R}$, one obtains a linear map $\Gamma_g : \mathcal{T} \to \mathcal{T}$ by $\Gamma_g \tau = (\mathrm{id} \otimes g)\Delta\tau$. Denoting by $\mathcal{G}_+$ the set of multiplicative linear functionals g on $\mathcal{T}_+$, we then set

$$\mathcal{G}_+ = \{g \in \mathcal{T}_+^* : \ g(\tau\bar{\tau}) = g(\tau)g(\bar{\tau}), \ \forall \tau, \bar{\tau} \in \mathcal{T}_+\} \ ,$$

and we define $\mathcal{G}$ by

$$\mathcal{G} = \{\Gamma_g : g \in \mathcal{G}_+\} \ . \tag{2.5}$$

It is straightforward to verify that $\mathcal{G}$ has the desired properties, including the fact that its elements respect the product structure of $\mathcal{T}$ in the sense that $\Gamma(\tau\bar{\tau}) = \Gamma\tau \cdot \Gamma\bar{\tau}$. Furthermore, $\mathcal{G}$ preserves not only $\mathcal{T}_{\text{ex}}$, but also $\mathcal{T}$, so that it also serves as the structure group for $\mathcal{T}$.

2.2 Admissible models

We now start to introduce a class of admissible models for our regularity structure. As in [Hai14b], we fix a truncation K of the heat kernel which coincides with it near the origin and annihilates polynomials of degree up to 3. The existence of such a kernel K is easy to show, and can be found, for example, in [Hai14b, Sec. 5.1].

We let $\mathcal{D}'$ denote the space of Schwartz distributions on $\mathbf{R}^{1+3}$ and $\mathcal{L}(\mathcal{T},\mathcal{D}')$ the space of linear maps from $\mathcal{T}$ to $\mathcal{D}'$. Furthermore, for any test function $\varphi : \mathbf{R}^{1+3} \to \mathbf{R}$, any $z \in \mathbf{R}^{1+3}$ and $\lambda \in \mathbf{R}^+$, we use φ_z^λ to denote $\varphi_z^\lambda(z') = \lambda^{-5}\varphi((t'-t)\lambda^{-2}, (x'-x)\lambda^{-1})$. We also write $\mathcal{B}$ for the set of smooth functions $\varphi\colon \mathbf{R}^4 \to \mathbf{R}$ that are compactly supported in $\{|z| \leq 1\}$ whose derivatives up to order three (including the value of the function) are uniformly bounded by 1.

Recall that a model for $(\mathcal{T},\mathcal{G})$ consists of a pair (Π, F) of functions

$$\begin{aligned} \Pi : \mathbf{R}^{1+3} &\to \mathcal{L}(\mathcal{T},\mathcal{D}') &\qquad F : \mathbf{R}^{1+3} &\to \mathcal{G} \\ z &\mapsto \Pi_z & z &\mapsto F_z \end{aligned}$$

satisfying the identity

$$\Pi_z F_z^{-1} = \Pi_{\bar z} F_{\bar z}^{-1}, \qquad \forall z, \bar z, \tag{2.6}$$

as well as the bounds

$$|(\Pi_z \tau)(\varphi_z^\lambda)| \lesssim \lambda^{|\tau|}, \qquad |\Gamma_{z,\bar z}\tau|_\sigma \lesssim |z-\bar z|^{|\tau|-|\sigma|} \tag{2.7}$$

uniformly over all $\varphi \in \mathcal{B}$, all space-time points $z, \bar z$ in compact domains and every $\tau \in \mathcal{W}$, where we used the shorthand $\Gamma_{z,\bar z} = F_z^{-1} \circ F_{\bar z}$, and the proportionality constant depends on the compact domain $\mathfrak{K}$. We will write f_z for the element in $\mathcal{G}_+$ such that $F_z = \Gamma_{f_z}$. We will give explicit expressions for f_z, and will write the notation (Π, f) for a model frequently. We also write $|\tau|_\sigma$ for the norm of the component of τ in $\mathcal{T}_\sigma$ (the precise choice of norm does not matter since these spaces are all finite-dimensional). We define the norm of a model $\mathfrak{M} = (\Pi, f)$ to be the smallest constant that makes both bounds in (2.7) to hold, and denote it by $|\!|\!|\mathfrak{M}|\!|\!|_{\mathfrak{K}}$. Since in most of the situations, F is completely determined by Π, we sometimes also write $|\!|\!|\Pi|\!|\!|$ instead of $|\!|\!|\mathfrak{M}|\!|\!|$, and we omit the domain $\mathfrak{K}$ wherever no confusion may arise. With these notations, we can define what we mean by an *admissible* model.

Definition 2.2. *A model* (Π, f) *is admissible if for every multi-index k, one has*

$$(\Pi_z X^k)(\bar z) = (\bar z - z)^k, \qquad f_z(X^k) = (-z)^k \tag{2.8}$$

and for every $\tau \in \mathcal{W}$ with $\mathcal{I}(\tau) \in \mathcal{T}$, one has

$$\begin{aligned} f_z(\mathscr{J}_\ell \tau) &= -\int D^\ell K(z-\bar z)(\Pi_z\tau)(d\bar z), \qquad |\ell| < |\tau| + 2 \\ \Pi_z \mathcal{I}(\tau)(\bar z) &= (K * \Pi_z \tau)(\bar z) + \sum_\ell \frac{(\bar z - z)^\ell}{\ell!} \cdot f_z(\mathscr{J}_\ell \tau). \end{aligned} \tag{2.9}$$

Here, we set $\mathscr{J}_\ell \tau = 0$ if $|\ell| \geq |\tau| + 2$, so the sum is always finite.

See [Hai14b] for the correct way of interpreting these identities in case Π_z contains distributions that are not functions.

2.3 Canonical lift to $\mathcal{T}_{\mathrm{ex}}$

Given any *smooth* space-time function $\widehat{\xi}$ and any real number ϵ, there is a canonical way to build an admissible model $\mathscr{L}_\epsilon(\widehat{\xi}) = (\Pi^\epsilon, f^\epsilon)$ for the regularity structure $(\mathcal{T}_{\mathrm{ex}}, \mathcal{G})$ as follows. We first set

$$(\Pi_z^\epsilon \Xi)(\bar{z}) = \widehat{\xi}(\bar{z}),$$

independent of ϵ and the base point z. We then define $\Pi_z^\epsilon \tau$ recursively for other $\tau \in \mathcal{W}$ by (2.9) as well as the identities

$$(\Pi_z^\epsilon \tau\bar{\tau})(\bar{z}) = (\Pi_z^\epsilon \tau)(\bar{z}) \cdot (\Pi_z^\epsilon \bar{\tau})(\bar{z}) \tag{2.10}$$

and

$$\begin{aligned} f_z^\epsilon(\mathscr{E}_\ell^\beta \tau) &= -\epsilon^\beta (D^\ell(\Pi_z^\epsilon \tau))(z), \\ (\Pi_z \mathcal{E}^\beta \tau)(\bar{z}) &= \epsilon^\beta (\Pi_z^\epsilon \tau)(\bar{z}) + \sum_\ell \frac{(\bar{z}-z)^\ell}{\ell!} \cdot f_z^\epsilon(\mathscr{E}_\ell^\beta \tau). \end{aligned} \tag{2.11}$$

Here, we again adopt the convention $\mathscr{E}_\ell^\beta(\tau) = 0$ if $|\ell| \geq \beta + |\tau|$. This construction makes sense only when $\Pi_z \tau$ is sufficiently regular, and this is indeed the case if $\widehat{\xi}$ is smooth. We then have the following fact, the proof of which can be found in [HQ15].

Proposition 2.3. *Let $\widehat{\xi}$ be a smooth space-time function, and $\epsilon \geq 0$. Then, the canonical model $\mathscr{L}_\epsilon(\widehat{\xi}) = (\Pi^\epsilon, f^\epsilon)$ defined via the identities* (2.8) – (2.11) *is an admissible model.*

Later on, we will consider the situation where $\widehat{\xi} = \xi_\epsilon$, a regularised version of the space-time white noise ξ, so we are led to the canonical model $\mathscr{L}_\epsilon(\xi_\epsilon)$. However, it is important to note that at this stage nothing forces the values of the two ϵ's to be identical: it is perfectly legitimate to consider the model $\mathscr{L}_\epsilon(\xi_\delta)$ for any pair of (ϵ, δ).

Also, one would like the linear map $\mathcal{E}^\beta$ to represent the multiplication by ϵ^β. This is however not quite true in view of (2.11), and it suggests that we should introduce a new map $\widehat{\mathcal{E}}^\beta$ on the $\mathcal{D}^\gamma$ space of modelled distributions (see Section 3 in [Hai14b] for a definition) by

$$(\widehat{\mathcal{E}}^\beta U)(z) = \mathcal{E}^\beta U(z) - \sum_\ell \frac{X^\ell}{\ell!} f_z(\mathscr{E}_\ell^\beta(U(z))). \tag{2.12}$$

Then, as long as the model is admissible and satisfies (2.11), the map $\widehat{\mathcal{E}}^\beta$ does indeed represent multiplication by ϵ^β in the sense that $\mathcal{R}\hat{\mathcal{E}}^\beta U = \epsilon^\beta \mathcal{R}U$ for $\mathcal{R}$ the reconstruction operator.

2.4 Renormalisation

The aim of this section is to build a group $\mathfrak{R}$ of transformations that we can use to "renormalise" our models. It is crucial for our purpose that such a renormalisation procedure satisfies the following three properties:

1. $\mathfrak{R}$ acts on the space $\mathscr{M}$ of *admissible* models.

2. $\mathfrak{R}$ is sufficiently rich so that one can find elements $M_\epsilon \in \mathfrak{R}$ such that $M_\epsilon \mathscr{L}_\epsilon(\xi_\epsilon)$ converges to a limit in $\mathscr{M}$, where $\mathscr{L}_\epsilon$ denotes the "canonical lift" of the regularised noise ξ_ϵ.

3. Solving the fixed point problem (3.1) for a model of the type $M\mathscr{L}_\epsilon(\eta)$ for a smooth space-time function η and $M \in \mathfrak{R}$ leads to the solution of a modified PDE.

The transformations $M \in \mathfrak{R}$ we consider here will be composed by two linear maps M_0 and M^{Wick} on $\mathcal{T}_{\text{ex}}$. The map M^{Wick} encodes "Wick renormalisation", while M_0 has the interpretation as mass renormalisation in the quantum field theory. From now on, we will use the shorthand $\Psi = \mathcal{I}(\Xi)$. We start with the standard Wick renormalisation map M^{Wick} on $\mathcal{T}_{\text{ex}}$. Define the generator L^{Wick} by

$$L^{\text{Wick}}\Xi = L^{\text{Wick}}X^k = 0, \qquad L^{\text{Wick}}\Psi^k = \binom{k}{2}\Psi^{k-2},$$

and extend this to the whole of $\mathcal{T}_{\text{ex}}$ by

$$L^{\text{Wick}}(\tau\mathcal{I}(\bar\tau)) = L^{\text{Wick}}(\tau)\mathcal{I}(\bar\tau) + \tau\mathcal{I}(L^{\text{Wick}}\bar\tau) ,$$

for $\bar\tau \neq \Xi$, as well as

$$L^{\text{Wick}}\mathcal{I}(\tau) = \mathcal{I}(L^{\text{Wick}}\tau), \quad L^{\text{Wick}}(\mathcal{E}^\beta\tau) = \mathcal{E}^\beta(L^{\text{Wick}}\tau), \quad L^{\text{Wick}}(X^k\tau) = X^k L^{\text{Wick}}\tau .$$

The map $M^{\text{Wick}} : \mathcal{T}_{\text{ex}} \to \mathcal{T}_{\text{ex}}$ is then defined by

$$M^{\text{Wick}} = \exp(-C_1 L^{\text{Wick}}) . \tag{2.13}$$

The definition of L^{Wick} ensures that M^{Wick} commutes with X^k as well as with the abstract integration maps $\mathcal{I}$ and $\mathcal{E}^\beta$. M^{Wick} has the interpretation as Wick renormalisation in the sense that

$$M^{\text{Wick}}\Psi^k = C_1^{\frac{k}{2}} H_k(\Psi/\sqrt{C_1}) =: H_k(\Psi; C_1) , \tag{2.14}$$

where $H_k(\cdot)$ is the k-th Hermite polynomial whose leading order coefficient is normalised to 1. For example, we have

$$H_1(\Psi;C_1) = \Psi, \qquad H_2(\Psi;C_1) = \Psi^2 - C_1, \qquad H_3(\Psi;C_1) = \Psi^3 - 3C_1\Psi.$$

Note that although we will always consider the case where $C_1 \geq 0$, the above expression $H_k(\Psi;C_1)$ actually does not require C_1 to be positive.

We now describe the effect of M^{Wick} on the canonical model (Π, f). Following [Hai14b, HQ15], for the map M^{Wick} defined above, there is a unique pair of linear maps

$$\Delta^{\text{Wick}} : \mathcal{T}_{\text{ex}} \to \mathcal{T}_{\text{ex}} \otimes \mathcal{T}_+, \qquad \widehat{M}^{\text{Wick}} : \mathcal{T}_+ \to \mathcal{T}_+$$

satisfying

$$\begin{aligned}
\widehat{M}^{\text{Wick}}\mathscr{J}_\ell &= \mathcal{M}(\mathscr{J}_\ell \otimes \mathrm{id})\Delta^{\text{Wick}},\\
\widehat{M}^{\text{Wick}}\mathscr{E}_\ell^\beta &= \mathcal{M}(\mathscr{E}_\ell^\beta \otimes \mathrm{id})\Delta^{\text{Wick}},\\
(\mathrm{id}\otimes\mathcal{M})(\Delta\otimes\mathrm{id})\Delta^{\text{Wick}} &= (M^{\text{Wick}} \otimes \widehat{M}^{\text{Wick}})\Delta,\\
\widehat{M}^{\text{Wick}}(\tau_1\tau_2) &= (\widehat{M}^{\text{Wick}}\tau_1)(\widehat{M}^{\text{Wick}}\tau_2), \qquad \widehat{M}^{\text{Wick}}X^k = X^k,
\end{aligned} \tag{2.15}$$

where $\mathcal{M} : \mathcal{T}_+ \to \mathcal{T}_+$ denotes the multiplication in the Hopf algebra $\mathcal{T}_+$. As in [HQ15, Sec. 5], one can verify that both $\widehat{M}^{\text{Wick}}$ and Δ^{Wick} have the relevant triangular structure, so that if, given an admissible model (Π, f), we define $(\Pi^{\text{Wick}}, f^{\text{Wick}})$ by

$$\Pi_z^{\text{Wick}}\tau = (\Pi_z \otimes f_z)\Delta^{\text{Wick}}\tau, \qquad f_z^{\text{Wick}}(\sigma) = f_z(\widehat{M}^{\text{Wick}}\sigma) , \tag{2.16}$$

then $(\Pi^{\text{Wick}}, f^{\text{Wick}})$ is again an admissible model. Furthermore, as a consequence of the second identity in (2.15) and the fact that M^{Wick} commutes with $\mathcal{E}^\beta$, if (Π, f) satisfies (2.11) for some ϵ, then so does $(\Pi^{\text{Wick}}, f^{\text{Wick}})$.

We now turn to describing the map M_0. For $n \geq 2$, we define linear maps L_n and L'_n on $\mathcal{T}_{\text{ex}}$ by setting

$$\begin{aligned}
L_n : \qquad & \mathcal{E}^{\frac{n}{2}-1}(\Psi^n\mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^n)) \mapsto n! \cdot \mathbf{1},\\
& \mathcal{E}^{\frac{n}{2}-1}(\Psi^n\mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^{n+1})) \mapsto (n+1)! \cdot \Psi,\\
& \mathcal{E}^{\frac{n}{2}-\frac{1}{2}}(\Psi^n\mathcal{I}(\mathcal{E}^{\frac{n}{2}-\frac{3}{2}}\Psi^n)) \mapsto n! \cdot \mathbf{1}, \quad n \geq 3,\\
& \mathcal{E}^{\frac{n}{2}-\frac{1}{2}}(\Psi^{n+1}\mathcal{I}(\mathcal{E}^{\frac{n}{2}-\frac{3}{2}}\Psi^n)) \mapsto (n+1)! \cdot \Psi, \quad n \geq 3,\\
L'_n : \qquad & \mathcal{E}^{\frac{n}{2}-1}(\Psi^n\mathcal{I}(\mathcal{E}^{\frac{n}{2}-\frac{3}{2}}\Psi^n)) \mapsto n! \cdot 1, \quad n \geq 3,
\end{aligned}$$

(we use the convention $\mathcal{E}^0 = \text{id}$) and $L_n\tau = 0$, $L'_n\tau = 0$ for any other basis vector $\tau \in \mathcal{W}$. Given these maps, we then consider maps on $\mathcal{T}_{\text{ex}}$ of the form

$$M_0 := \exp\Big(-\sum_{n\geq 2} C_n L_n - \sum_{n\geq 2} C'_n L'_n\Big).$$

As we will see in (3.15), at the level of abstract equation, M_0 has the simple effect of adding a linear term to the right hand side of the equation. Actually, M_0 is equivalently given by

$$M_0 = \text{id} - \sum_{n\geq 2} C_n L_n - \sum_{n\geq 2} C'_n L'_n \;.$$

Furthermore, it commutes with $\mathcal{G}$ in the sense that $M_0\Gamma\tau = \Gamma M_0\tau$ for any $\tau \in \mathcal{T}$ and $\Gamma \in \mathcal{G}$. As a consequence, given an admissible model $(\bar{\Pi}, \bar{f})$, if we set

$$\bar{\Pi}_z^{M_0}\tau := \bar{\Pi}_z M_0\tau, \qquad \bar{f}_z^{M_0}\sigma = \bar{f}_z(\sigma) , \tag{2.17}$$

then $(\bar{\Pi}^{M_0}, \bar{f}^{M_0})$ is also an admissible model. Given $M = (M_0, M^{\text{Wick}})$ with M_0 and M^{Wick} as above, we then define the renormalised model (Π^M, f^M) by

$$\Pi_z^M\tau = (\Pi_z \otimes f_z)\Delta^{\text{Wick}}(M_0\tau), \qquad f_z^M(\sigma) = f_z^{\text{Wick}}(\widehat{M}^{\text{Wick}}\sigma). \tag{2.18}$$

Remark 2.4. Note that although in many cases one has $(\Pi_z^M\tau)(z) = (\Pi_z M\tau)(z)$, this is in general not true. For example, for $\tau = \mathcal{E}\Psi^4$, we have $(\Pi_z^M\tau)(z) = \epsilon\big(\widehat{\xi}^4(z) - 6C_1\widehat{\xi}^2(z) + 3C_1^2\big)$, while $(\Pi_z M\tau)(z) = \epsilon\big(\widehat{\xi}^4(z) - 6C_1\widehat{\xi}^2(z)\big)$.

3 Abstract fixed point problem

In this section, we translate (1.2) into a fixed point problem in a suitable space of modelled distributions. It is natural to consider the fixed point problem

$$\Phi = \mathcal{P}\mathbf{1}_+\Big(\Xi - \sum_{j=4}^{m} \lambda_j \mathcal{Q}_{\le 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}(\mathcal{Q}_{\le 0}(\Phi^j)) - \sum_{j=0}^{3} \lambda_j \mathcal{Q}_{\le 0}(\Phi^j)\Big) + \widehat{P}u_0, \tag{3.1}$$

where $\mathcal{Q}_{\le\alpha}$ denote the projection onto the subspace $\bigoplus_{\beta\le\alpha}\mathcal{T}_\beta$ in $\mathcal{T}_{\text{ex}}$, $\widehat{P}u_0$ is the canonical lift of the solution to the deterministic heat equation with initial data u_0 to the regularity structure, and $\mathcal{P}$ denotes the operator given by

$$\mathcal{P} = \mathcal{K} + \widehat{R}\mathcal{R},$$

where $\mathcal{K}$ is the abstract integration operator defined from the truncated heat kernel K as in [Hai14b, Sec.4], $\mathcal{R}$ is the reconstruction operator, and $\widehat{R}u$ is the Taylor expansion of the smooth function $(P - K) * u$ up to order γ.

To solve such a fixed point problem, at first glance, it seems that one can simply follow the procedure in [Hai14b, Sec. 7] to obtain a unique solution to (3.1) in a space $\mathcal{D}^{\gamma,\eta}$ as in [Hai14b, Sec. 6] for suitable γ and η. Unfortunately, as in [HQ15], this argument only works for sufficiently regular initial data (it needs to be "almost continuous" for large values of m). Since the dynamical Φ^4_3 model only has regularity $\mathcal{C}^\eta$ for $\eta < -\frac{1}{2}$, this would prevent us from using a continuation argument to control the convergence of our models on any fixed time interval. In addition, such a continuation argument also requires one to be able to evaluate the reconstructed solution $\mathcal{R}\Phi$ in a suitable space of distributions at any fixed time. However, as one can easily see, the solution to (3.1) contains the term $\Psi = \mathcal{I}(\Xi)$ which has negative homogeneity, and a priori there is no clear way to give meaning to $\mathcal{R}\Psi$ at any fixed time t. The second issue is not a serious problem here since, for the natural model constructed from space-time white noise, $\mathcal{R}\Psi$ can indeed be regarded as a continuous function (in time) in a suitable space of distributions. (See for example [Hai14b, EJS13].)

To resolve the first issue, we introduce ϵ-dependent norms to enforce suitable control on both our admissible models and the initial condition as $\epsilon \to 0$. In a way, this allows us to "trade" the singularities near $t = 0$ and at small scales for powers of ϵ.

In what comes below, we will mainly follow [HQ15] to build such weighted spaces. It turns out that the algebraic structure of these spaces are essentially the same as those in introduced in [HQ15], except that the values of γ and η are different. We will therefore mostly give statements and refer to [HQ15] for detailed proofs.

3.1 The ϵ-dependent spaces and models

Below, we use φ to denote a space-time test function belonging to $\mathcal{B}$, ϕ to denote such a test function that furthermore integrates to 0, and ψ to denote a test function that annihilates affine functions of the spatial variables.

Recall that our definition of an admissible model in the previous section does not specify any relationship between its actions on τ and $\mathcal{E}^\beta(\tau)$. In order to formulate the cancellation of the singularity in time by the small parameter ϵ in the limiting process

$\epsilon \to 0$, we introduce the space of models $\mathscr{M}_\epsilon$ which consists of all *admissible models* (Π, f) with the further restriction that

$$\begin{aligned} |f_z(\mathscr{E}_\ell^\beta(\tau))| &\lesssim \epsilon^{\beta-|\ell|+|\tau|}, \qquad \tau \in \mathcal{W}\,, \\ |\langle \Pi_z \tau, \psi_z^\lambda \rangle| &\lesssim \lambda^\zeta \cdot \epsilon^{|\tau|-\zeta}, \qquad \tau \in \mathcal{U},\ \zeta = \frac{6}{5}\,. \end{aligned} \tag{3.2}$$

Here, all the bounds are to hold uniformly over all space-time points z in compact sets, all $\lambda \in (0, \epsilon)$ and all test functions $\psi \in \mathcal{B}$ that annihilate affine functions. We also require that, for some sufficiently large $\eta < -\frac{1}{2}$ (to be fixed below),

$$\sup_{t\in[0,1]} \|\Pi_0 \Psi(t,\cdot)\|_{\mathcal{C}^\eta} < +\infty.$$

We will verify later in Section 5 that the models considered in this article do indeed belong to $\mathscr{M}_\epsilon$ with uniform controls as $\epsilon \to 0$.

We let $\|\Pi\|_\epsilon$ denote the smallest proportionality constant for both bounds in (3.2), and define a "norm" on $\mathscr{M}_\epsilon$ by

$$|\!|\!|\Pi|\!|\!|_\epsilon := |\!|\!|\Pi|\!|\!| + \|\Pi\|_\epsilon + \sup_{t\in[0,1]} \|\Pi_0 \Psi(t,\cdot)\|_{\mathcal{C}^\eta},$$

where $|\!|\!|\Pi|\!|\!|$ is the usual "norm" on admissible models introduced in Section 2.2. Again, these norms all depend on the compact set $\mathfrak{K}$ where the supremum of z is taken over, which we have omitted for notational simplicity.

Remark 3.1. This is of course an abuse of notation since $|\!|\!|\Pi|\!|\!|$ and $\|\Pi\|_\epsilon$ both depend not only on Π but also on F, and F can in general not be recovered uniquely from Π and the knowledge that the model is admissible (unlike in the situations considered in [Hai14b]). We chose to nevertheless keep this notation for the sake of conciseness. Also, the norm $|\!|\!|\Pi|\!|\!|_\epsilon$ depends not only on ϵ but also on η. Since we will fix the value $\eta < -\frac{1}{2}$ below, we omit η in the notation.

We compare two models in $\mathscr{M}_\epsilon$ by

$$|\!|\!|\Pi;\bar\Pi|\!|\!|_\epsilon = |\!|\!|\Pi;\bar\Pi|\!|\!| + \|\Pi - \bar\Pi\|_\epsilon + \sup_{t\in[0,1]} \|\Pi_0 \Psi(t,\cdot) - \bar\Pi_0 \Psi(t,\cdot)\|_{\mathcal{C}^\eta}.$$

We also denote by $\mathscr{M}_0$ the class of admissible models such that $f_z(\mathscr{E}_\ell^\beta(\tau)) \equiv 0$. It is natural to compare two elements $(\Pi^{(\epsilon)}, \Gamma^{(\epsilon)}) \in \mathscr{M}_\epsilon$ and $(\Pi, \Gamma) \in \mathscr{M}_0$ by

$$|\!|\!|\Pi^{(\epsilon)};\Pi|\!|\!|_{\epsilon;0} = |\!|\!|\Pi^{(\epsilon)};\Pi|\!|\!| + \|\Pi^{(\epsilon)}\|_\epsilon + \sup_{t\in[0,1]} \|\Pi_0^{(\epsilon)} \Psi(t,\cdot) - \Pi_0 \Psi(t,\cdot)\|_{\mathcal{C}^\eta}.$$

Note that $\mathscr{M}_\epsilon$ and $\mathscr{M}_{\epsilon'}$ consists of exactly the same class of models for each $\epsilon, \epsilon' > 0$, but with different scales on their norms. The point here is that we will consider models with $|\!|\!|\Pi^{(\epsilon)};\Pi|\!|\!|_{\epsilon;0} \to 0$ for some limiting model Π. We first give a useful lemma.

Lemma 3.2. *There exists $C > 0$ such that, for $\Pi \in \mathscr{M}_\epsilon$ and $\tau \in \mathcal{U}$, we have*

$$|\langle \Pi_z \tau, \varphi_z^\lambda \rangle| < C\|\Pi\|_\epsilon \epsilon^{|\tau|}\,, \qquad |\langle \Pi_z \tau, \phi_z^\lambda \rangle| < C\|\Pi\|_\epsilon \lambda \epsilon^{|\tau|-1}\,, \tag{3.3}$$

uniformly over all $\lambda < \epsilon < 1$, all space-time points z in compact sets and all test functions $\varphi, \phi \in \mathcal{B}$ with the further restriction that ϕ integrates to 0.

Proof. We first prove the second bound. Let ϕ be a test function integrating to 0, let $\lambda \in (0,1]$, and let N be the integer such that

$$\lambda \cdot 2^N \leq \epsilon < \lambda \cdot 2^{N+1}. \tag{3.4}$$

We then write ϕ_z^λ as a telescope sum by

$$\phi_z^\lambda = \sum_{k=0}^{N-1} (2^{-k}\phi_z^{\lambda\cdot 2^k} - 2^{-(k+1)}\phi_z^{\lambda\cdot 2^{k+1}}) + 2^{-N}\cdot \phi_z^{\lambda\cdot 2^N} =: \sum_{k=0}^{N-1} \delta\phi_z^{\lambda,k} + 2^{-N}\cdot\phi_z^{\lambda\cdot 2^N}\ .$$

For each k appearing in this sum, $\delta\phi_z^{\lambda,k}$ is localised at scale $\lambda\cdot 2^k < \epsilon$ and integrates to 0 since the function ϕ does. Furthermore, the factor 2^{-k} is chosen such that the integral of $2^{-k}\phi_z^{\lambda\cdot 2^k}$ against linear functions does not depend on k, so that $\delta\phi_z^{\lambda,k}$ annihilates all affine functions. Thus, we can use the second bound in (3.2) to deduce that for each k, we have

$$|\langle \Pi_z\tau, \delta\phi_z^{\lambda,k}\rangle| < C\|\Pi\|_\epsilon 2^{-k}(\lambda 2^k)^\zeta \cdot \epsilon^{|\tau|-\zeta}\ .$$

Summing over k from 0 to $N-1$, and using the fact that $\zeta > 1$ and $\lambda\cdot 2^N \sim \epsilon$, we conclude that $\sum_{k=0}^{N-1}\langle\Pi_z\tau, \delta\phi_z^{\lambda,k}\rangle < C\|\Pi\|_\epsilon \lambda\epsilon^{|\tau|-1}$. The same bound holds for the term $2^{-N}\cdot\phi_z^{\lambda\cdot 2^N}$ as a direct consequence of (2.7), so we obtain the second bound in (3.3).

To prove the first one, fix a test function φ, and write it as

$$\varphi_z^\lambda = \sum_{k=0}^{N-1}(\varphi_z^{\lambda\cdot 2^k} - \varphi_z^{\lambda\cdot 2^{k+1}}) + \varphi_z^{\lambda\cdot 2^N}. \tag{3.5}$$

This time, each function in the parenthesis integrates to 0 so we can use the second bound just proved above, and the first one follows easily. $\square$

We now turn to dealing with the irregularity of the initial data. At this point, our definitions start to differ from those in [HQ15] in order to encode the regularities of terms in (3.1). We first introduce a new space for the initial condition u_0.

Definition 3.3. *Let $\gamma \in (1,2)$, $\eta < 0$ and $\epsilon > 0$. The space $\mathcal{C}_\epsilon^{\gamma,\eta}$ consists of $\mathcal{C}^\gamma$ functions $f\colon \mathbf{R}^3 \to \mathbf{R}$ with norm*

$$\|f\|_{\gamma,\eta;\epsilon} = \|f^{(\epsilon)}\|_{\mathcal{C}^\eta} + \epsilon^{-\eta}\|f^{(\epsilon)}\|_\infty + \epsilon^{\gamma-\eta}\sup_{|x-y|<\epsilon}\frac{|Df^{(\epsilon)}(x) - Df^{(\epsilon)}(y)|}{|x-y|^{\gamma-1}}. \tag{3.6}$$

Furthermore, we set $\mathcal{C}_0^{\gamma,\eta} = \mathcal{C}^\eta$. The distance between two elements $f^{(\epsilon)} \in \mathcal{C}_\epsilon^{\gamma,\eta}$ and $f \in \mathcal{C}^\eta$ is defined by

$$\|f^{(\epsilon)}; f\|_{\gamma,\eta;\epsilon} = \|f^{(\epsilon)} - f\|_{\mathcal{C}^\eta} + \epsilon^{-\eta}\|f^{(\epsilon)}\|_\infty + \epsilon^{\gamma-\eta}\sup_{|x-y|<\epsilon}\frac{|Df^{(\epsilon)}(x) - Df^{(\epsilon)}(y)|}{|x-y|^{\gamma-1}}. \tag{3.7}$$

The reason we do not include a bound on $\|Df^{(\epsilon)}\|_\infty$ on the right hand side is that such a bound follows the bounds on $\|f^{(\epsilon)}\|_\infty$ and $\|Df^{(\epsilon)}\|_{\mathcal{C}^{\gamma-1}}$. More precisely, one has

Lemma 3.4. *There exists a constant C such that, for every $f^{(\epsilon)} \in \mathcal{C}_\epsilon^{\gamma,\eta}$ one has*

$$\|Df^{(\epsilon)}\|_\infty < C\|f^{(\epsilon)}\|_{\gamma,\eta;\epsilon} \cdot \epsilon^{\eta-1} \,. \tag{3.8}$$

Proof. The proof is straightforward and we leave it as an exercise. □

One should think of functions in $\mathcal{C}_\epsilon^{\gamma,\eta}$ as behaving like elements of $\mathcal{C}^\eta$ at large scales, while being of class $\mathcal{C}^\gamma$ at small scales, with ϵ determining where the cutoff between "small" and "large" lies. The reason why only $f^{(\epsilon)}$ appears in the last two terms of (3.7) is that these two quantities are not even finite for general $f \in \mathcal{C}^\eta$.

Following [HQ15, Sec. 3.5], we define $\mathcal{D}^{\gamma,\eta}$ space to be the set of functions U taking values in $\mathcal{T}$ with norm

$$\begin{aligned}\|U\|_{\gamma,\eta} :=& \sup_z \sup_{|\tau|<\gamma} |U(z)|_\tau + \sup_z \sup_{|\tau|<\gamma} \frac{|U(z)|_\tau}{\sqrt{|t|}^{(\eta-|\tau|)\wedge 0}} \\ &+ \sup_{|z-z'|<\sqrt{|t|\wedge|t'|}} \sup_{|\tau|<\gamma} \frac{|U(z)-\Gamma_{z,z'}U(z')|_\tau}{|z-z'|^{\gamma-|\tau|}\sqrt{|t|\wedge|t'|}^{\eta-\gamma}} .\end{aligned}$$

Note that this definition is slightly different from the original one in [Hai14b] in the sense that it allows $U(z)$ to have components in $\mathcal{T}_{\geq\gamma}$. We now introduce the weighted spaces $\mathcal{D}_\epsilon^{\gamma,\eta}$ that are suitable for our fixed point problem.

Definition 3.5. *For each ϵ, γ, η, and each model $(\Pi, \Gamma) \in \mathscr{M}_\epsilon$, the space $\mathcal{D}_\epsilon^{\gamma,\eta}$ consists of modelled distributions U with norm given by*

$$\|U\|_{\gamma,\eta;\epsilon} = \|U\|_{\gamma,\eta} + \sup_z \sup_\tau \frac{|U(z)|_\tau}{\epsilon^{(\eta-|\tau|)\wedge 0}} + \sup_{(z,z')\in D_\epsilon} \sup_{|\tau|<\gamma} \frac{|U(z)-\Gamma_{z,z'}U(z')|_\tau}{|z-z'|^{\gamma-|\tau|}\epsilon^{\eta-\gamma}} \,.$$

Here, the supremum is taken over all space-time points $(z, z') \in D_\epsilon$, defined by

$$D_\epsilon = \{(z,z') : |z-z'| < \epsilon \wedge \sqrt{|t|\wedge|t'|}\} \,,$$

where $z = (t, x)$, $z' = (t', x')$, and $\|\cdot\|_{\gamma,\eta}$ is the norm of the usual $\mathcal{D}^{\gamma,\eta}$ spaces introduced in [Hai14b, Sec. 6].

In short, the above definition says that modelled distributions U in $\mathcal{D}_\epsilon^{\gamma,\eta}$ satisfy the bounds

$$\begin{aligned}|U(z)|_\tau &\lesssim (\epsilon + \sqrt{|t|})^{(\eta-|\tau|)\wedge 0}, \\ |U(z)-\Gamma_{z,z'}U(z')|_\tau &\lesssim |z-z'|^{\gamma-|\tau|}(\epsilon+\sqrt{|t|\wedge|t'|})^{\eta-\gamma}.\end{aligned}$$

Note that $\mathcal{D}_\epsilon^{\gamma,\eta}$ is a linear space once the model is fixed, and so the distance between two elements can be simply compared by $\|U-\bar{U}\|_{\gamma,\eta;\epsilon}$. Also, in all the cases we consider below, η is always smaller than the regularity of the sector in consideration. Thus, we will have $\eta < |\tau|$, and can simply replace $(\eta - |\tau|) \wedge 0$ by $\eta - |\tau|$ in all the situations below. Similar as before, we compare two elements $U^{(\epsilon)} \in \mathcal{D}_\epsilon^{\gamma,\eta}$ and $U \in \mathcal{D}^{\gamma,\eta}$ by

$$\begin{aligned}\|U^{(\epsilon)}; U\|_{\gamma,\eta;\epsilon} =& \|U^{(\epsilon)}; U\|_{\gamma,\eta} + \sup_z \sup_\tau \frac{|U^{(\epsilon)}(z)|_\tau}{\epsilon^{(\eta-|\tau|)\wedge 0}} \\ &+ \sup_{(z,z')\in D_\epsilon} \sup_{|\tau|<\gamma} \frac{|U^{(\epsilon)}(z)-\Gamma_{z,z'}U^{(\epsilon)}(z')|}{|z-z'|^{\gamma-|\tau|}\epsilon^{\eta-\gamma}} .\end{aligned}$$

The reason why only $U^{(\epsilon)}$ appears on the latter two terms on the right hand side above is the same as before: these quantities are in general not finite for $U \in \mathcal{D}^{\gamma,\eta}$. The main motivation for the introduction of these ϵ-dependent spaces is that they contain the solution to the heat equation with initial condition in $\mathcal{C}_\epsilon^{\gamma,\eta}$, with bounds independent of ϵ. This is the content of the following proposition, the proof of which is identical to that of [HQ15, Prop. 4.7], so we do not repeat the details here.

Proposition 3.6. *Let $\eta < 0$, $\gamma \in (1,2)$, $\epsilon \in (0,1]$, and $u \in \mathcal{C}_\epsilon^{\gamma,\eta}$. Let $\widehat{P}u$ denote the canonical lift of the harmonic extension of u via its truncated Taylor expansion of order γ. Then, $\widehat{P}u \in \mathcal{D}_\epsilon^{\gamma,\eta}$ and one has*

$$\|\widehat{P}u\|_{\gamma,\eta;\epsilon} < C\|u\|_{\gamma,\eta;\epsilon}.$$

Furthermore, if $u^{(\epsilon)} \in \mathcal{C}_\epsilon^{\gamma,\eta}$ and $u \in \mathcal{C}^\eta$, then one has

$$\|\widehat{P}u^{(\epsilon)}; \widehat{P}u\|_{\gamma,\eta;\epsilon} < C\|u^{(\epsilon)}; u\|_{\gamma,\eta;\epsilon}.$$

The following proposition will be needed later when we continue local solutions up to their (potential) explosion time. It says that the initial data of the restarted solution still belongs to the $\mathcal{C}_\epsilon^{\gamma,\eta}$ space with norms uniform in ϵ.

Proposition 3.7. *Let $\gamma \in (1, \frac{6}{5})$, and $\eta \in (-\frac{m+1}{2m+1}, -\frac{1}{2})$. Let $(\Pi^\epsilon, f^\epsilon) \in \mathscr{M}_\epsilon$. Let $\mathcal{U}$ be a sector of the regularity structure as defined in* (2.2). *If $\mathcal{R}^\epsilon$ is the associated reconstruction map for $\mathcal{D}_\epsilon^{\gamma,\eta}(\mathcal{U})$ and $U^{(\epsilon)} \in \mathcal{D}_\epsilon^{\gamma,\eta}(\mathcal{U})$ is the abstract solution to* (3.1), *then for every $t > 0$, $u_t^{(\epsilon)} := \mathcal{R}^\epsilon U^{(\epsilon)}(t,\cdot)$ belongs to $\mathcal{C}_\epsilon^{\gamma,\eta}$ with*

$$\|u_t^{(\epsilon)}\|_{\gamma,\eta;\epsilon} < C\|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon}\|\!|\Pi^{(\epsilon)}|\!\|_\epsilon.$$

If (Π, f) is another such model with reconstruction operator $\mathcal{R}$, and $U \in \mathcal{D}^{\gamma,\eta}$ solves (3.1) *based on Π, then $u_t := \mathcal{R}U(t,\cdot)$ belongs to $\mathcal{C}^\eta$ and one has*

$$\|u_t^{(\epsilon)}; u_t\|_{\gamma,\eta;\epsilon} \lesssim \|U^{(\epsilon)}; U\|_{\gamma,\eta;\epsilon}(\|\!|\Pi|\!\| + \|\!|\Pi^\epsilon|\!\|_\epsilon) + \|\!|\Pi^\epsilon; \Pi|\!\|_{\epsilon,0}(\|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon} + \|U\|_{\gamma,\eta}) .$$

Proof. We first prove the first claim of the proposition. For that, we bound separately the three terms appearing in the definition (3.6) of the spaces $\mathcal{C}_\epsilon^{\gamma,\eta}$. We first notice that any solution $U^{(\epsilon)}$ to (3.1) is necessarily of the form

$$U^{(\epsilon)}(z) = \Psi + V^{(\epsilon)}(z) .$$

Since the structure group acts trivially on Ψ, the constant function Ψ belongs to all spaces $\mathcal{D}_\epsilon^{\gamma,\eta}$, so that if $U^{(\epsilon)} \in \mathcal{D}_\epsilon^{\gamma,\eta}$, then so does $V^{(\epsilon)}$. Since, in the above decomposition, $V^{(\epsilon)}(z)$ belongs to the linear span of $\{\mathbf{1}\} \cup \{\tau : |\tau| > 0\}$, the desired bound for $\|\mathcal{R}^\epsilon V(t,\cdot)\|_{\mathcal{C}^\eta}$ follows from [Hai14b, Prop. 3.28]. Regarding the term Ψ, one has $\mathcal{R}^\epsilon\Psi = \Pi_0^\epsilon\Psi$ so that, by the definition of $\mathscr{M}_\epsilon$, we have

$$\sup_{t\in[0,1]} \|(\mathcal{R}^\epsilon\Psi)(t,\cdot)\|_{\mathcal{C}^\eta} < C\|\!|\Pi^\epsilon|\!\|_\epsilon,$$

and the required bound for $\|u_t^{(\epsilon)}\|_{\mathcal{C}^\eta}$ thus follows.

For the remaining two terms on the right hand side of (3.6), we will prove a stronger bound by showing $u^{(\epsilon)} = \mathcal{R}^\epsilon U^{(\epsilon)}$ is a space-time function with desired regularity, rather just being a function in space for fixed time.

For the second term, since the lowest homogeneity in $\mathcal{U}$ is $-\frac{1}{2} - \kappa$, an application of the reconstruction theorem together with Lemma 3.2 gives

$$\sup_{\lambda<\epsilon} \sup_z \sup_{\varphi\in\mathcal{B}} |\langle u^{(\epsilon)}, \varphi_z^\lambda \rangle| < C \|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon} \|\!|\Pi^\epsilon|\!\|_\epsilon \cdot \epsilon^{-\frac{1}{2}-\kappa}.$$

On the other hand, it follows directly from the definition of a model that

$$\sup_{\lambda\geq\epsilon} \sup_z \sup_{\varphi\in\mathcal{B}} \lambda^{\frac{1}{2}+\kappa} |\langle u^{(\epsilon)}, \varphi_z^\lambda \rangle| < C \|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon} \|\!|\Pi^\epsilon|\!\|_\epsilon.$$

Combining the above two bounds and using the fact that κ is arbitrarily small so that $\eta < -\frac{1}{2} - \kappa$, we conclude that $u^{(\epsilon)}$ is a continuous function with

$$\epsilon^{-\eta} \|u^{(\epsilon)}\|_\infty < C\epsilon^{-\eta-\frac{1}{2}-\kappa} \cdot \|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon} \|\!|\Pi^\epsilon|\!\|_\epsilon.$$

We now turn to the third term on the right hand side of (3.6). In order to show $Du^{(\epsilon)} \in \mathcal{C}^{\gamma-1}$, we test it against test functions that integrate to 0. Using the definition of the distributional derivative and then the triangle inequality, we get

$$\lambda^{1-\gamma} |\langle Du^{(\epsilon)}, \phi_z^\lambda \rangle| \leq \lambda^{-\gamma} |\langle \Pi_z^\epsilon U^{(\epsilon)}(z), (D\phi)_z^\lambda \rangle| + \lambda^{-\gamma} |\langle u^{(\epsilon)} - \Pi_z^\epsilon U^{(\epsilon)}(z), (D\phi)_z^\lambda \rangle|.$$

It follows from the reconstruction theorem that the second term on the right hand side above is uniformly bounded by a constant. For the first term, since the assumption that ϕ integrates to 0 implies $D\phi$ annihilates affine functions, we can use the second bound in (3.2) to obtain

$$\lambda^{-\gamma} |\langle \Pi_z^\epsilon U^{(\epsilon)}(z), (D\phi)_z^\lambda \rangle| < C \|U^{(\epsilon)}\|_{\gamma,\eta;\epsilon} \|\!|\Pi^\epsilon|\!\|_\epsilon \cdot \lambda^{\zeta-\gamma} \epsilon^{-\frac{1}{2}-\kappa-\zeta},$$

where we again used the fact that the lowest homogeneity in $\mathcal{U}$ is $-\frac{1}{2} - \kappa$. The desired bound then follows immediately.

For the second claim, the only problematic term is $\|u_t^{(\epsilon)}; u_t\|_{\mathcal{C}^\eta}$, but again the desired bound for this term follows in the same way as $\|u_t^{(\epsilon)}\|_{\mathcal{C}^\eta}$. □

Before we proceed to further properties of the $\mathcal{D}_\epsilon^{\gamma,\eta}$ spaces, we first make a few remarks about these spaces and our notation.

- The set D_ϵ in Definition 3.5 is taken to be $\{|z - z'| < \epsilon \wedge \sqrt{|t| \wedge |t'|}\}$. This is sufficient since for the pairs (z, z') such that $\epsilon \leq |z - z'| < \sqrt{|t| \wedge |t'|}$, we have $\epsilon + \sqrt{|t| \wedge |t'|} < 2\sqrt{|t| \wedge |t'|}$, so the bound on the last term in Definition 3.5 follows automatically from the bound on $\|\cdot\|_{\gamma,\eta}$.

- We use the notation $\|F - \bar{F}\|_{\gamma,\eta;\epsilon}$ to compare two functions in the *same* $\mathcal{D}_\epsilon^{\gamma,\eta}$ space with the *same* underlying model. On the other hand, whenever we write $\|F; \bar{F}\|_{\gamma,\eta;\epsilon}$, it should be understood that we are comparing $F \in \mathcal{D}_\epsilon^{\gamma,\eta}$ with $\bar{F} \in \mathcal{D}^{\gamma,\eta}$, typically based on a different model. As we will never compare two functions belonging to $\mathcal{D}_\epsilon^{\gamma,\eta}$ spaces with the same ϵ but different underlying models, these notations are sufficient.

It turns out that these spaces behave as expected under multiplication and action of $\widehat{\mathcal{E}}^k$ and $\mathcal{P}$. We state a few of the properties we will be using later; most of the proofs can be found in [HQ15].

Proposition 3.8. *Let $U_i \in \mathcal{D}_\epsilon^{\gamma_i,\eta_i}(V^{(i)})$ for $i = 1, 2$, where $V^{(1)}$ and $V^{(2)}$ are sectors of respective regularities α_1 and α_2. If*

$$\gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1), \qquad \eta = (\eta_1 + \alpha_2) \wedge (\eta_2 + \alpha_1) \wedge (\eta_1 + \eta_2),$$

then their pointwise product $U = U_1 U_2$ is in $\mathcal{D}_\epsilon^{\gamma,\eta}$ with

$$\|U\|_{\gamma,\eta;\epsilon} < C\|U_1\|_{\gamma_1,\eta_1;\epsilon}\|U_2\|_{\gamma_2,\eta_2;\epsilon}.$$

Furthermore, if $\bar{U}_i \in \mathcal{D}_\epsilon^{\gamma_i,\eta_i}$, then $\bar{U} = \bar{U}_1\bar{U}_2 \in \mathcal{D}^{\gamma,\eta}$ with the same η, γ as above, and we have

$$\|U; \bar{U}\|_{\gamma,\eta;\epsilon} < C(\|U_1; \bar{U}_1\|_{\gamma,\eta;\epsilon} + \|U_2; \bar{U}_2\|_{\gamma,\eta;\epsilon} + \|\Gamma - \bar{\Gamma}\|),$$

where C is proportional to $\sum_i(\|U_i\| + \|\bar{U}_i\|) + \|\Gamma\| + \|\bar{\Gamma}\|$.

Proposition 3.9. *Let $U \in \mathcal{D}_\epsilon^{\gamma,\eta}$ with $\eta \leq \gamma$. If $\alpha \geq \gamma$, then $\mathcal{Q}_{\leq\alpha}U \in \mathcal{D}_\epsilon^{\gamma,\eta}$ with*

$$\|\mathcal{Q}_{\leq\alpha}U\|_{\gamma,\eta;\epsilon} \lesssim \|U\|_{\gamma,\eta;\epsilon}.$$

Proposition 3.10. *Let $U \in \mathcal{D}_\epsilon^{\gamma,\eta}$ with $\gamma \in (-\beta, 1-\beta)$. Then $\widehat{\mathcal{E}}^\beta U \in \mathcal{D}_\epsilon^{\gamma',\eta'}$ with*

$$\gamma' = (\gamma + \beta) \wedge \inf_{|\tau|<\gamma}(\gamma - |\tau|), \qquad \eta' = \eta + \beta,$$

and we have the bound

$$\|\widehat{\mathcal{E}}^\beta U\|_{\gamma',\eta';\epsilon} < C(1 + \|\Pi\|_\epsilon)\|U\|_{\gamma,\eta;\epsilon}.$$

In addition, if $\bar{U} \in \mathcal{D}^{\gamma,\eta}$ with model $\bar{\Pi} \in \mathscr{M}_0$, we have

$$\|\widehat{\mathcal{E}}^\beta U; \widehat{\mathcal{E}}^\beta \bar{U}\|_{\gamma',\eta';\epsilon} < C(1 + \|\Pi\|_\epsilon)(\|U; \bar{U}\|_{\gamma,\eta;\epsilon} + |||\Pi; \bar{\Pi}|||_{\epsilon;0})$$

with the same γ' and η'.

Proposition 3.11. *Let $U \in \mathcal{D}_\epsilon^{\gamma,\eta}(V)$, where V is a sector of regularity α with $-2 < \eta < \gamma \wedge \alpha$. Then, provided that γ and η are not integers, we have $\mathcal{P}U \in \mathcal{D}_\epsilon^{\bar{\gamma},\bar{\eta}}$ with $\bar{\gamma} = \gamma + 2$ and $\bar{\eta} = \eta + 2$, and we have the bound*

$$\|\mathcal{P}U\|_{\bar{\gamma},\bar{\eta};\epsilon} < C\|U\|_{\gamma,\eta;\epsilon}.$$

Furthermore, if $\bar{U} \in \mathcal{D}_0^{\gamma,\eta}$, then we also have

$$\|\mathcal{P}U; \mathcal{P}\bar{U}\|_{\bar{\gamma},\bar{\eta};\epsilon} < C(\|U; \bar{U}\|_{\gamma,\eta;\epsilon} + |||\Pi, \bar{\Pi}|||_\epsilon).$$

3.2 Solution to the fixed point problem and convergence

We now have all the ingredients in place to build our solution with uniform (in ϵ) bounds in suitable $\mathcal{D}_\epsilon^{\gamma,\eta}$ spaces. The equation we consider is of a general form that it sufficiently flexible to cover all the concrete cases to be considered later. We first show the existence and uniqueness of local solutions.

Theorem 3.12. *Let $m \geq 1$, $\gamma \in (1, \frac{6}{5})$, $\eta \in (-\frac{m+1}{2m+1}, -\frac{1}{2})$, and $\kappa > 0$ be sufficiently small. Let $\phi_0 \in \mathcal{C}_\epsilon^{\gamma,\eta}$, and consider the equation*

$$\Phi = \mathcal{P}\mathbf{1}_+\bigg(\Xi - \sum_{j=4}^{m} \lambda_j \mathcal{Q}_{\leq 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}(\mathcal{Q}_{\leq 0}\Phi^j) - \sum_{j=0}^{3} \lambda_j \mathcal{Q}_{\leq 0}(\Phi^j)\bigg) + \widehat{P}\phi_0. \tag{3.9}$$

Then, for every sufficiently small ϵ and every model in $\mathscr{M}_\epsilon$, there exists $T > 0$ such that the equation (3.9) *has a unique solution in $\mathcal{D}_\epsilon^{\gamma,\eta}$ up to time T. Moreover, T can be chosen uniformly over any fixed bounded set of initial data in $\mathcal{C}_\epsilon^{\gamma,\eta}$, any bounded set of models in $\mathscr{M}_\epsilon$, any bounded set of parameters $\lambda_j^{(\epsilon)}$, and all sufficiently small ϵ.*

Let $\phi_0^{(\epsilon)}$ be a sequence of elements in $\mathcal{C}_\epsilon^{\gamma,\eta}$ such that $\|\phi_0^{(\epsilon)}; \phi_0\|_{\gamma,\eta;\epsilon} \to 0$ for some $\phi_0 \in \mathcal{C}^\eta$, $\Pi^\epsilon \in \mathscr{M}_\epsilon$, $\Pi \in \mathscr{M}_0$ be models such that $|||\Pi^\epsilon; \Pi|||_{\epsilon;0} \to 0$, and let $\lambda_j^{(\epsilon)} \to \lambda_j$ for each j. If $\Phi \in \mathcal{D}^{\gamma,\eta}$ solves the fixed point problem (3.9)*with model Π, initial data ϕ_0 and coefficients λ_j up to time T, then for all small enough ϵ, there is a unique solution $\Phi^{(\epsilon)} \in \mathcal{D}_\epsilon^{\gamma,\eta}$ to* (3.9) *with $\Pi^\epsilon, \phi_0^{(\epsilon)}$ and $\lambda_j^{(\epsilon)}$ up to the same time T, and we have*

$$\lim_{\epsilon \to 0} |||\Phi^{(\epsilon)}; \Phi|||_{\gamma,\eta;\epsilon} \to 0.$$

Proof. We first prove that the fixed point problem (3.9) can be solved in $\mathcal{D}_\epsilon^{\gamma,\eta}$ with local existence time uniform in ϵ. Let $\mathcal{M}_T^{(\epsilon)}$ denote the map

$$\mathcal{M}_T^{(\epsilon)}(\Phi) = \mathcal{P}\mathbf{1}_+\bigg(\Xi - \sum_{j=4}^{m} \lambda_j \mathcal{Q}_{\leq 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}(\mathcal{Q}_{\leq 0}\Phi^j) - \sum_{j=0}^{3} \lambda_j \mathcal{Q}_{\leq 0}(\Phi^j)\bigg) + \widehat{P}\phi_0. \tag{3.10}$$

We will show that, for T sufficiently small, $\mathcal{M}_T^{(\epsilon)}$ is a contraction mapping a centered ball in $\mathcal{D}_\epsilon^{\gamma,\eta}$ of a large enough radius Ω into a ball of radius $\frac{\Omega}{2}$. In what follows, we will omit the subscript T and simply write the map as $\mathcal{M}^{(\epsilon)}$.

We first show that $\mathcal{M}^{(\epsilon)}$ maps $\mathcal{D}_\epsilon^{\gamma,\eta}$ into itself. By Proposition 3.6, we have $\widehat{P}\phi_0^{(\epsilon)} \in \mathcal{D}_\epsilon^{\gamma,\eta}$. In addition, the noise term $\mathcal{P}\mathbf{1}_+\Xi$ also belongs to $\mathcal{D}_\epsilon^{\gamma,\eta}$. As for the non-linearity, if $j \leq 3$, it is straightforward to see that $\Phi^j \in \mathcal{D}_\epsilon^{\delta,3\eta}$ for some positive δ. We can choose δ small enough so that there is no basis vectors with homogeneity between 0 and 2δ, and Proposition 3.9 implies that $\mathcal{Q}_{\leq 0}(\Phi^j) = \mathcal{Q}_{\leq 2\delta}(\Phi^j) \in \mathcal{D}_\epsilon^{\delta,3\eta}$. It is then an immediate application of Proposition 3.11 to see that the map

$$\Phi \mapsto \mathcal{P}\mathbf{1}_+\bigg(\sum_{j=0}^{2} \lambda_j \mathcal{Q}_{\leq 0}(\Phi^j)\bigg)$$

is locally Lipschitz from $\mathcal{D}_\epsilon^{\gamma,\eta}$ into itself with a Lipschitz constant bounded by $(T + \epsilon)^\theta$ for some positive θ.

We now turn to the nonlinear term $\mathcal{P}\mathbf{1}_+(\mathcal{Q}_{\leq 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}(\Phi^j))$ for $j \geq 4$. Let

$$\gamma_j = \gamma - \frac{j-1}{2} - (j-1)\kappa, \qquad \eta_j = j\eta, \qquad \bar{\eta}_j = j\eta + \frac{j-3}{2}.$$

Then by Propositions 3.8 and 3.9, we have $\mathcal{Q}_{\leq 0}(\Phi^j) \in \mathcal{D}_\epsilon^{\gamma_j,\eta_j}$ with

$$\|\mathcal{Q}_{\leq 0}(\Phi^j)\|_{\gamma_j,\eta_j;\epsilon} < C\|\Phi\|_{\gamma,\eta;\epsilon}^j.$$

The assumption $\gamma > 1$ implies $\gamma_j > -\frac{j-3}{2}$ if κ is sufficiently small, so applying Proposition 3.10 with $\beta = \frac{j-3}{2}$, we know that there exists $\delta > 0$ such that $\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}(\Phi^j) \in \mathcal{D}_\epsilon^{\delta,\bar{\eta}_j}$ with

$$\|\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}(\Phi^j)\|_{\delta,\bar{\eta}_j;\epsilon} < C(1+\|\Pi\|_\epsilon)\|\Phi\|_{\gamma,\eta;\epsilon}^j.$$

Similar as before, we can again choose δ to be small enough so that $\mathcal{Q}_{\leq 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}(\Phi^j) = \mathcal{Q}_{\leq 2\delta}\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}(\Phi^j)$ also belongs to $\mathcal{D}_\epsilon^{\delta,\bar{\eta}_j}$ with the same bound. Since $\bar{\eta}_j > -2$, an application of Proposition 3.11 implies that there exists $\theta > 0$ such that

$$\|\mathcal{P}\mathbf{1}_+\widehat{\mathcal{E}}^{\frac{j-3}{2}}(\mathcal{Q}_{\leq 0}(\Phi^j))\|_{\gamma,\eta;\epsilon} < C(T+\epsilon)^\theta(1+\|\Pi\|_\epsilon)\|\Phi\|_{\gamma,\eta;\epsilon}^{j+2}.$$

This shows $\mathcal{M}^{(\epsilon)}$ indeed maps $\mathcal{D}_\epsilon^{\gamma,\eta}$ into itself. In particular, if Λ is big enough with

$$\|\Phi\|_{\gamma,\eta;\epsilon} < \Lambda, \qquad \|u_0^{(\epsilon)}\|_{\gamma,\eta;\epsilon} < \frac{\Lambda}{C},$$

then we can choose T small enough depending on Λ, $\|\Pi\|_\epsilon$ and $\lambda_j^{(\epsilon)}$'s only such that

$$\|\mathcal{M}^{(\epsilon)}(\Phi)\|_{\gamma,\eta;\epsilon} < \frac{\Lambda}{2}.$$

In order to show $\mathcal{M}^{(\epsilon)}$ is also a contraction for small T, we first note that since there is only one model concerned in $\mathscr{M}_\epsilon$, we can simply compare the difference $\mathcal{M}^{(\epsilon)}(\Phi) - \mathcal{M}^{(\epsilon)}(\tilde{\Phi})$ for two elements $\Phi, \tilde{\Phi} \in \mathcal{D}_\epsilon^{\gamma,\eta}$. In fact, we have

$$\begin{aligned}\mathcal{M}^{(\epsilon)}(\Phi) - \mathcal{M}^{(\epsilon)}(\tilde{\Phi}) = &-\sum_{j=4}^{m}\sum_{k=0}^{j-1}\lambda_j\mathcal{P}\mathbf{1}_+\mathcal{Q}_{\leq 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}\mathcal{Q}_{\leq 0}((\Phi-\tilde{\Phi})\Phi^{j-1-k}\tilde{\Phi}^k)\\ &- \mathcal{Q}_{\leq 0}(\Phi-\tilde{\Phi})(\lambda_3(\Phi^2+\Phi\tilde{\Phi}+\tilde{\Phi}^2)+\lambda_2(\Phi+\tilde{\Phi})+\lambda_1).\end{aligned}$$

By linearity, $\Phi - \tilde{\Phi} \in \mathcal{D}_\epsilon^{\gamma,\eta}$, so all the bounds obtained above also apply for $\mathcal{M}^{(\epsilon)}(\Phi) - \mathcal{M}^{(\epsilon)}(\tilde{\Phi})$ except that one power of $\|\Phi\|_{\gamma,\eta;\epsilon}$ is replaced by $\|\Phi-\tilde{\Phi}\|_{\gamma,\eta;\epsilon}$. Thus, we get

$$\begin{aligned}&\|\mathcal{M}^{(\epsilon)}(\Phi) - \mathcal{M}^{(\epsilon)}(\tilde{\Phi})\|_{\gamma,\eta;\epsilon}\\ &< C(T+\epsilon)^\theta\|\Phi-\tilde{\Phi}\|_{\gamma,\eta;\epsilon}(1+\|\Pi\|_\epsilon)(1+\|\Phi\|_{\gamma,\eta;\epsilon}+\|\tilde{\Phi}\|_{\gamma,\eta;\epsilon})^{m-1}.\end{aligned}$$

Again, if we restrict ourselves to centered balls with radius Λ in $\mathcal{D}_\epsilon^{\gamma,\eta}$, then as soon as we choose

$$(T+\epsilon)^\theta < \frac{1}{C(1+\|\Pi\|_\epsilon)(1+2\Lambda)^{m-1}}, \tag{3.11}$$

the map $\mathcal{M}^{(\epsilon)} = \mathcal{M}_T^{(\epsilon)}$ is a contraction and there is a unique solution to (3.9). This is possible for all small ϵ. In addition, it is clear that if the coefficients $\lambda_j^{(\epsilon)}$'s, the norms $\|\Pi\|_\epsilon$ and $\|u_0^{(\epsilon)}\|_{\gamma,\eta;\epsilon}$ are uniformly bounded as $\epsilon \to 0$, then this short existence time T could be chosen independent of ϵ provided ϵ is small enough.

We now turn to the second part of the theorem, namely the convergence of local solutions $\Phi^{(\epsilon)}$ to Φ up to the time T when Φ is defined. By the arguments above, there exists a time $S < T$ such that (3.9) has a fixed point solution $\Phi^{(\epsilon)}$ in $\mathcal{D}_\epsilon^{\gamma,\eta}$ up to time S for all small ϵ. We first show the convergence of $\Phi^{(\epsilon)}$ to Φ up to time S, and iterate the relative bounds to get existence and convergence to time T.

Let $\mathcal{M}^{(\epsilon)} : \mathcal{D}_\epsilon^{\gamma,\eta} \to \mathcal{D}_\epsilon^{\gamma,\eta}$ denote the map

$$\mathcal{M} : \Phi \mapsto \mathcal{P}\mathbf{1}_+\bigg(\Xi - \sum_{j=4}^{m} \lambda_j^{(\epsilon)} \mathcal{Q}_{\le 0}\widehat{\mathcal{E}}^{\frac{j-3}{2}}(\mathcal{Q}_{\le 0}(\Phi^j)) - \sum_{j=0}^{3} \lambda_j^{(\epsilon)} \mathcal{Q}_{\le 0}(\Phi^j)\bigg) + \widehat{P}\phi_0^{(\epsilon)}.$$

up to time S, and $\mathcal{M} : \mathcal{D}^{\gamma,\eta} \to \mathcal{D}^{\gamma,\eta}$ be the map of the same form except that $\lambda_j^{(\epsilon)}$ and $\phi_0^{(\epsilon)}$ are replaced by λ_j and ϕ_0. Following the same line of argument as in the proof for the first half, we have

$$\begin{aligned} &\|\mathcal{M}^{(\epsilon)}(\Phi^{(\epsilon)}); \mathcal{M}(\Phi)\|_{\gamma,\eta;\epsilon} \\ &\lesssim (S+\epsilon)^\theta \|\Phi^{(\epsilon)}; \Phi\|_{\gamma,\eta;\epsilon} + \sup_j |\lambda_j^{(\epsilon)} - \lambda_j| + \|\!|\Pi^{(\epsilon)}; \Pi|\!\|_{\epsilon,0} + \|\phi_0^{(\epsilon)}; \phi_0\|_{\gamma,\eta;\epsilon}, \end{aligned}$$

where the proportionality constant depends on the norm of the relevant models, the size of the ball in $\mathcal{D}_\epsilon^{\gamma,\eta}$, the initial data and the coefficients. For small enough S, using the knowledge that $\Phi^{(\epsilon)}$ and Φ are the fixed points in $\mathcal{D}_\epsilon^{\gamma,\eta}$ and $\mathcal{D}_0^{\gamma,\eta}$ respectively, we easily get

$$\|\Phi^{(\epsilon)}; \Phi\|_{\gamma,\eta;\epsilon} \lesssim \sup_j |\lambda_j^{(\epsilon)} - \lambda_j| + \|\!|\Pi^{(\epsilon)}; \Pi|\!\|_{\epsilon,0} + \|u_0^{(\epsilon)}; u_0\|_{\gamma,\eta;\epsilon}. \tag{3.12}$$

This gives the desired convergence of $\|\Phi^{(\epsilon)}; \Phi\|_{\gamma,\eta;\epsilon}$ to 0 up to time S. We now need to extend the solutions to time T, up to when the solution Φ to (3.9) is defined with model $\Pi \in \mathscr{M}_0$. It suffices to have bounds for $\mathcal{R}^{(\epsilon)}\Phi^{(\epsilon)}(t,\cdot)$ and $\mathcal{R}^{(\epsilon)}\Phi^{(\epsilon)}(t,\cdot) - (\mathcal{R}\Phi)(t,\cdot)$ in $\mathcal{C}_\epsilon^{\gamma,\eta}$ so that we can restart the solution from time t. In fact, these are precisely what we obtained in Proposition 3.7. Thus, one could iterate (3.12) up to time T, and this completes the proof. $\square$

3.3 Renormalised equation

We now turn to studying the effect of the renormalisation maps defined in Section 2.4 on the solutions to the fixed point problem (3.9). For simplicity, we write

$$\mathcal{F} := \sum_{j=3}^{m} \lambda_j \mathcal{E}^{\frac{j-3}{2}} \Psi^j,$$

and, for $n \ge 1$, we define the n-th 'derivative' of $\mathcal{F}$ to be

$$\mathcal{F}^{(n)} := \sum_{j=3}^{m} j(j-1)\cdots(j-n+1)\lambda_j \mathcal{E}^{\frac{j-3}{2}} \Psi^{j-n}.$$

If $(\bar{\Pi}, \bar{f})$ is an admissible model and $\gamma \in (1, \frac{6}{5})$, then the solution to the fixed point problem (3.9) in $\mathcal{D}_\epsilon^{\gamma,\eta}$ necessarily has the form

$$\Phi = \Psi + \varphi \cdot \mathbf{1} - \mathcal{I}(\mathcal{F}) - \lambda_2 \mathcal{I}(\Psi^2) - \varphi \cdot \mathcal{I}(\mathcal{F}') + \varphi' \cdot X = \Psi + U, \tag{3.13}$$

where U denotes the part that contains all basis vectors except Ψ. Therefore, the right hand side of (3.9) (including all terms with homogeneities up to 0) can be written as

$$\begin{aligned}
\mathcal{H}(z) &= \Xi - \sum_{j=4}^{m} \lambda_j \widehat{\mathcal{E}}^{\frac{j-3}{2}} \Phi^j - \sum_{j=0}^{3} \lambda_j \Phi^j \qquad (3.14)\\
&= \Xi - \mathcal{F} - \lambda_2 \Psi^2 - \varphi \cdot \mathcal{F}' - \frac{1}{2}\varphi^2 \cdot \mathcal{F}'' - (\lambda_1 + 2\varphi\lambda_2)\Psi + \mathcal{F}'\mathcal{I}(\mathcal{F})\\
&\quad + \lambda_2 \mathcal{F}'\mathcal{I}(\Psi^2) + \varphi \cdot \mathcal{F}'\mathcal{I}(\mathcal{F}') + \varphi \cdot \mathcal{F}''\mathcal{I}(\mathcal{F}) + 2\lambda_2 \Psi \mathcal{I}(\mathcal{F}) - \varphi' \mathcal{F}' X - \frac{1}{6}\varphi^3 \cdot \mathcal{F}'''\\
&\quad + \Bigg(\lambda_0 + \lambda_1 \varphi + \lambda_2 \varphi^2 + \sum_{j \geq 4} \sum_{n=4}^{j} \binom{j}{n} \varphi^n \overline{f}_z(\mathscr{E}_0^{\frac{j-3}{2}}(\Psi^{j-n} U(z)^n)) \Bigg) \cdot \mathbf{1}.
\end{aligned}$$

We then have the following theorem, the proof of which is essentially the same as that in [HQ15, Sec. 5], so we omit the details here.

Theorem 3.13. *Let $\phi_0 \in \mathcal{C}^1$, $\epsilon \geq 0$, and $\widehat{\xi}$ be a smooth space-time function. Let $(\Pi, f) = \mathscr{L}_\epsilon(\widehat{\xi})$ be the canonical model as in Section 2, $M = (M_0, M^{\text{Wick}})$, and $(\Pi^M, f^M) = M\mathscr{L}_\epsilon(\widehat{\xi})$ be the renormalised model described in Section 2.4. If $\Phi \in \mathcal{D}_\epsilon^{\gamma,\eta}$ is the local solution to the fixed point problem* (3.9) *for the model (Π^M, f^M), then the function $u = \mathcal{R}^M \Phi$ is the classical solution to the PDE*

$$\partial_t u = \Delta u - \sum_{j=4}^{m} \lambda_j \epsilon^{\frac{j-3}{2}} H_j(u; C_1) - \sum_{j=0}^{3} \lambda_j H_j(u; C_1) - (Cu + C' + 6\lambda_2\lambda_3 C_2) + \widehat{\xi} \tag{3.15}$$

with initial data ϕ_0, and the constants C and C' are given by

$$\begin{aligned}
C &= \sum_{n=2}^{m-1} (n+1)^2 n! \cdot \lambda_{n+1}^2 C_n + \sum_{n=3}^{m-2} (n+2)! \cdot \lambda_n \lambda_{n+2} C_n\,,\\
C' &= \sum_{n=3}^{m-1} (n+1)! \cdot \lambda_n \lambda_{n+1} C'_n\,.
\end{aligned} \tag{3.16}$$

4 Convergence of the renormalised models

In this section, we will show how to choose the correct constants so that the action of the renormalisation maps built in Section 2.4 on the canonical model yields convergence to a limit, and we will also identify the limiting model. The identification of the limiting equation will be given in Section 5.

4.1 Main statement and convergence criterion

Let ξ denote space-time white noise on $\mathbf{R} \times \mathbf{T}^3$. Fix a smooth compactly supported function $\rho : \mathbf{R}^{1+3} \to \mathbf{R}$ integrating to 1, and set

$$\rho_\epsilon(t,x) = \epsilon^{-5}\rho(t/\epsilon^2, x/\epsilon), \qquad \xi_\epsilon = \rho_\epsilon * \xi, \tag{4.1}$$

where '$*$' denotes space-time convolution. Then, the correlation of ξ_ϵ is

$$\mathbf{E}\xi_\epsilon(s,x)\xi_\epsilon(t,y) = \int_{\mathbf{R}}\int_{\mathbf{T}^3} \rho_\epsilon(s-u, x-z)\rho_\epsilon(t-u, y-z)dzdu.$$

If the noise $\widehat{\xi}$ is obtained from the convolution of the space time white noise defined on $\mathbf{R} \times (\epsilon^{-1}\mathbf{T})^3$ with the mollifier ρ, then we actually have

$$\xi_\epsilon(t,x) \stackrel{\text{law}}{=} \epsilon^{-\frac{5}{2}}\widehat{\xi}(t/\epsilon^2, x/\epsilon).$$

From now on, we will always assume that the noise ξ_ϵ relates to ξ by (4.1). When we consider scale $\alpha < 1$ later, we simply replace ϵ by ϵ^α in that expression. We also let

$$K_\epsilon = K * \rho_\epsilon, \qquad G_\epsilon = K_\epsilon * K_\epsilon,$$

where K coincides the heat kernel in $\{|z| < 1\}$, has compact support, and annihilates polynomials up to degree 3, as introduced at the beginning of Section 2.2. We have the following easy proposition.

Proposition 4.1. *We have*

$$D^\ell K_\epsilon(z) \lesssim (|z| + \epsilon)^{-3-|\ell|}, \qquad G_\epsilon(z) \lesssim (|z| + \epsilon)^{-1},$$

uniformly over all $\epsilon < 1$ and space-time points z with $|z| < 1$.

Remark 4.2. Here, $\ell = (\ell_0, \ell_1, \ell_2, \ell_3)$ is a multi-index, and $|\ell| = 2\ell_0 + \sum_{i=1}^3 \ell_i$ reflects the parabolic scaling. In what follows, we will always use the notation $|\cdot|$ to denote the parabolic degree of such indices.

The main theorem of this section is the following.

Theorem 4.3. *Let $M_\epsilon \in \mathfrak{R}$ denote the renormalisation map*

$$M_\epsilon = \Big(\exp\big(-\sum_{n\geq 2} C_n^{(\epsilon)} L_n - \sum_{n\geq 3} C_n'^{(\epsilon)} L_n'\big), \ \ \exp\big(-C_1^{(\epsilon)} L_1\big)\Big),$$

with L_n and L_n' as in Section 2.4. Let $\mathscr{L}_\epsilon(\xi_\epsilon)$ be canonical lift of ξ_ϵ to the regularity structure $\mathcal{T}$ as in Section 2.3, and consider the sequence of models

$$\mathfrak{M}_\epsilon := M_\epsilon \mathscr{L}_\epsilon(\xi_\epsilon)\;.$$

Then, there exists a choice of constants $C_n^{(\epsilon)}$, $C_n'^{(\epsilon)}$, and a random model $\mathfrak{M} \in \mathscr{M}_0$ such that

$$|\!|\!|\mathfrak{M}_\epsilon; \mathfrak{M}|\!|\!|_{\epsilon,0} \to 0$$

in probability as $\epsilon \to 0$. Furthermore, the limiting model $\mathfrak{M} = (\widehat{\Pi}, \widehat{f})$ satisfies $\widehat{\Pi}_z\tau = 0$ for every z and every basis vector τ that contains an occurrence of $\mathcal{E}^\beta$ for some $\beta > 0$.

The readers may have already realised that with proper choices of $C_1^{(\epsilon)}$ and $C_2^{(\epsilon)}$, the action of the model $\mathfrak{M}_\epsilon$ on basis vectors without an appearance of $\mathcal{E}$ is exactly as those in the regularised Φ^4_3 equation (see [Hai14b, Section 10] for details). Thus, the action of the limiting model $\mathfrak{M}$ on those basis vectors is precisely the same as that of the limiting Φ^4_3 model.

However, the effect of the models $\mathfrak{M}_\epsilon$ on symbols that contain $\mathcal{E}$'s is more complicated. In order to prove Theorem 4.3, we first give a useful criterion for the convergence of models in $\mathscr{M}_\epsilon$. The proof of this criterion is essentially the same as Propositions 6.2 and 6.3 in [HQ15], so we only give the statement without proofs.

Proposition 4.4. *Let $(\mathcal{T},\mathcal{G})$ be the regularity structure given in Section 2, and consider a family of random models $(\widehat{\Pi}^\epsilon, \widehat{f}^\epsilon)$ in $\mathscr{M}_\epsilon$. Assume there exists $\theta > 0$ such that for every test function $\varphi \in \mathcal{B}$, every $\tau \in \mathcal{W}$ with $|\tau| < 0$, every space-time point z and every $\lambda \in (0,1)$, there exists a random variable $(\widehat{\Pi}_z \tau)(\varphi_z^\lambda)$ such that*

$$\mathbf{E}|(\widehat{\Pi}_z^\epsilon \tau)(\varphi_z^\lambda)|^2 \lesssim \lambda^{2|\tau|+\theta}, \qquad \mathbf{E}|(\widehat{\Pi}_z^\epsilon \tau - \widehat{\Pi}_z \tau)(\varphi_z^\lambda)|^2 \lesssim \epsilon^\theta \lambda^{2|\tau|+\theta}. \tag{4.2}$$

Assume furthermore that for every $\mathcal{E}^\beta(\tau) \in \mathcal{W}$ with $\beta + |\tau| > 0$, one has

$$\mathbf{E}|D^\ell \widehat{f}_z^\epsilon(\mathscr{E}_0^\beta \tau)| \lesssim \epsilon^{|\tau|+\beta-|\ell|+\theta} \tag{4.3}$$

for some positive θ, and that for any $\tau \in \mathcal{U}$, one has the bound

$$\mathbf{E}|(\widehat{\Pi}_z^\epsilon \tau)(\psi_z^\lambda)| \lesssim \lambda^{\zeta+\theta} \epsilon^{|\tau|-\zeta}, \quad \zeta = \frac{6}{5}, \tag{4.4}$$

for all test functions $\psi \in \mathcal{B}$ that annihilate affine functions, uniformly over $\lambda \in (0,\epsilon)$. Then, there exists a random model $(\widehat{\Pi}, \widehat{f}) \in \mathscr{M}_0$ such that $\|\!|\widehat{\Pi}^\epsilon, \widehat{\Pi}|\!\|_{\epsilon;0} \to 0$ in probability as $\epsilon \to 0$.

Remark 4.5. Later, we will consider $(\widehat{\Pi}^\epsilon, \widehat{f}^\epsilon) = M_\epsilon \mathscr{L}_\epsilon(\xi_\epsilon)$ as in Theorem 4.3 with proper renormalisation constants $C_j^{(\epsilon)}$'s defined in the next subsection. It is straightforward to see that they indeed belong to $\mathscr{M}_\epsilon$. For the limiting model $\mathfrak{M}$, its action on basis vectors without any appearance of $\mathcal{E}$ is exactly the same as in the standard Φ^4_3 equation (in fact, these are precisely the terms that appears in Φ^4_3). Its action on terms containing a factor of $\mathcal{E}^\beta$ will yield 0. Thus, in addition to (4.3), (4.4), it suffices to prove the second bound in (4.2) for τ containing at least one factor of $\mathcal{E}$, and with $\widehat{\Pi}_z \tau = 0$.

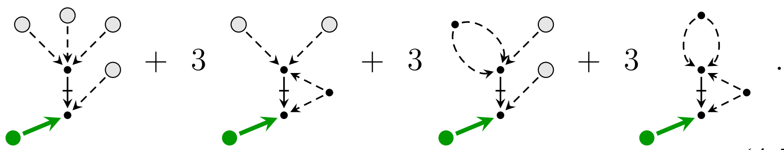
4.2 Graphical notations and preliminary bounds

The remainder of this section is devoted to the proof that the random models $M_\epsilon \mathscr{L}_\epsilon(\xi_\epsilon)$ as in Theorem 4.3 do indeed satisfy the convergence criterion of Proposition 4.4. Since we are in a translation invariant setting, it suffices bound the random variables $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ for various basis vectors τ. All these random variables belong to some finite order Wiener chaos. Following [HP14, HQ15], we use a graphical notation to represent the kernels for homogeneous Wiener chaos of finite order. Each node in the graph represents a space-time variable in $\mathbf{R}^{1+3}$: the special green node ● represents the origin, which is fixed, the nodes $\circ$ represent the arguments in the kernel representation for homogeneous Wiener chaos, and the remaining nodes $\bullet$ represent variables to be integrated out.

Each plain arrow $\longrightarrow$ represents the kernel $K(z' - z)$, where z and z' are starting and ending points of the arrow. A dotted arrow $\dashrightarrow$ represents the kernel K_ϵ with the same orientation as before, and a bold green arrow $\longrightarrow$ represents a generic test function in $\mathcal{B}$ rescaled by a factor λ. In addition, we use the barred arrow $\longrightarrow$ to represent a factor $K(z' - z) - K(-z)$, where as before z and z' denote starting and ending points of the arrow. Finally, a double barred arrow $\longrightarrow$ represents the factor $K(z' - z) - K(-z) - x' \cdot DK(-z)$, where $z = (t, x)$, $z' = (t', x')$, and the differentiation DK is with respect to space variable only.

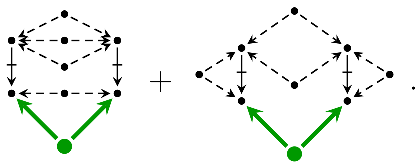
With these notations, it follows for example that for $\tau = \Psi\mathcal{I}(\Psi^3)$ and the canonical model $\Pi^\epsilon = \mathscr{L}_\epsilon(\xi_\epsilon)$, we have the expression

$$(\Pi_0^\epsilon \Psi\mathcal{I}(\Psi^3))(\varphi_0^\lambda) =$$



(4.5)

Here, the first term represents the component in the fourth homogeneous Wiener chaos (see [Nua06, Ch.1.1.2]), the next two terms represent the component in the second homogeneous chaos, and the last term is the component in the zeroth homogeneous chaos. The variance of the first two terms above, for example, are bounded (up to some constant multiple) by



(4.6)

To bound this and similar quantities, it is convenient to label the edges of the graph to reflect the singularity of the corresponding kernel, and to give a bound of the whole object in terms of simple power counting of the labels. For this purpose, and in order to be able to use the bounds obtained in [HQ15], we introduce *labelled graphs* to represent bounds for quantities like $\mathbf{E}|(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)|^2$.

In a labelled graph, each edge $e = (x_{v_-}, x_{v_+})$ comes with a pair of numbers $(a_e, r_e) \in \mathbf{R}^+ \times \mathbb{Z}$, and the orientation of the edge really matters only if $r_e > 0$. As before, edges e are associated to kernels J_e, with a_e measuring the singularity of the kernel in question in the sense that we assume that each J_e is compactly supported and satisfies a bound of the type

$$|D^k J_e(z)| \lesssim |z|^{-a_e - |k|} , \tag{4.7}$$

for every multiindex k. The precise factor represented by each edge then furthermore depends on the value r_e. If $r_e = 0$, then the corresponding edge simply represents a factor $\widehat{J}_e(x_{v_-}, x_{v_+}) = J_e(x_{v_+} - x_{v_-})$. We simply write a_e instead of $(a_e, 0)$ in this case.

If $r_e > 0$, then the corresponding edge represents a factor

$$\widehat{J}_e(x_{v_-}, x_{v_+}) = J_e(x_{v_+} - x_{v_-}) - \sum_{|k|_\mathfrak{s} < |r_e|} \frac{x_{v_+}^k}{k!} D^k J_e(-x_{v_-}). \tag{4.8}$$

On the other hand, if $r_e < 0$, then the edge corresponds to a factor $\widehat{J}_e(x_{v_-}, x_{v_+}) = (\mathscr{R} J_e)(x_{v_+} - x_{v_-})$, where $\mathscr{R} J_e$ denotes the *renormalised* distribution

$$(\mathscr{R} J_e)(\varphi) = \int J_e(x) \Big(\varphi(x) - \sum_{|k|_\mathfrak{s} < |r_e|} \frac{x^k}{k!} \varphi(0) \Big) \, dx \, . \tag{4.9}$$

Since we will always consider situations where no two edges with $r_e < 0$ meet and all J_e are smooth functions, the meaning of the factor $(\mathscr{R} J_e)(x_{v_+} - x_{v_-})$ is unambiguous.

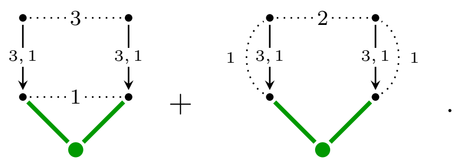
Unlike in [HQ15], each labelled graph does in our case represent a sequence of multiple integrals depending on a parameter $\epsilon \in (0, 1]$. To keep track of some of that dependency, we consider graphs with both 'plain' and 'dotted' edges. If an edge is plain, then the corresponding kernel J_e is allowed to depend on ϵ (to make that dependency clear we will also sometimes write $J_e^{(\epsilon)}$), but the bounds (4.7) are assumed to hold uniformly in $\epsilon \in (0, 1]$. If an edge is dotted however, then the corresponding kernel $J_e^{(\epsilon)}$ is assumed to satisfy the bound

$$|D^k J_e^{(\epsilon)}(z)| \lesssim (|z| + \epsilon)^{-a_e - |k|} \, ,$$

uniformly in $\epsilon \in (0, 1]$. There are two additional edges (in boldface) connecting to the origin that represent a factor $\varphi^\lambda(x_v, 0)$. The origin is denoted by $\{0\} \subset \mathcal{V}$, and we denote by $v_{\star,1}$ and $v_{\star,2}$ the two vertices that connect to 0 by the edges representing test functions. Finally, we set

$$\mathcal{V}_\star = \{0, v_{\star,1}, v_{\star,2}\}, \qquad \mathcal{V}_0 = \mathcal{V} \setminus \{0\}.$$

Thus, as a consequence of Proposition 4.1, the quantity in (4.6) can be represented by



With all these notations at hand, for a labelled graph $\mathcal{G}$ and the collection of kernels J_e, we let $I_\lambda^\mathcal{G}$ denote the number

$$I_\lambda^\mathcal{G} = \int_{(\mathbf{R}^4)^{\mathcal{V}_0}} \prod_{e \in \mathcal{E}} \widehat{J}_e(x_{e_-}, x_{e_+}) dx, \tag{4.10}$$

where 4 reflects the space-time dimension. In order to determine the right scale of the quantity $I_\lambda^\mathcal{G}$, we introduce some additional notations. For any subset $\bar{\mathcal{V}} \subset \mathcal{V}$, we let

$$\begin{aligned} \mathcal{E}^\uparrow(\bar{\mathcal{V}}) &= \{e \in \mathcal{E} : \mathcal{E} \cap \bar{\mathcal{V}} = e_-, r_e > 0\}; \\ \mathcal{E}^\downarrow(\bar{\mathcal{V}}) &= \{e \in \mathcal{E} : \mathcal{E} \cap \bar{\mathcal{V}} = e_+, r_e > 0\}; \\ \mathcal{E}_0(\bar{\mathcal{V}}) &= \{e \in \mathcal{E} : \mathcal{E} \cap \mathcal{V} = e\}; \\ \mathcal{E}(\bar{\mathcal{V}}) &= \{e \in \mathcal{E} : \mathcal{E} \cap \mathcal{V} \neq \phi\}. \end{aligned}$$

Consider a labelled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ satisfying the following properties.

Assumption 4.6. *The labelled graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ satisfies the following properties.*

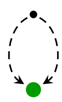
1. *For every edge $e \in \mathcal{E}$, one has $a_e + (r_e \wedge 0) < 5$;*
2. *For every subset $\bar{V} \subset \mathcal{V}$ of cardinality at least 3, one has*
$$\sum_{e \in \mathcal{E}_0(\bar{\mathcal{V}})} a_e < 5(|\bar{\mathcal{V}}| - 1);$$
3. *For every subset $\bar{\mathcal{V}} \subset \mathcal{V}$ containing 0 and of cardinality at least 2, one has*
$$\sum_{e \in \mathcal{E}_0(\mathcal{V})} a_e + \sum_{e \in \mathcal{E}^{\uparrow}(\bar{\mathcal{V}})} (a_e + r_e - 1) - \sum_{e \in \mathcal{E}^{\downarrow}(\bar{\mathcal{V}})} r_e < 5(|\bar{\mathcal{V}}| - 1);$$
4. *For every non-empty subset $\bar{\mathcal{V}} \subset \mathcal{V} \setminus \mathcal{V}_\star$, one has*
$$\sum_{e \in \mathcal{E}(\bar{\mathcal{V}}) \setminus \mathcal{E}^{\downarrow}(\bar{\mathcal{V}})} a_e + \sum_{e \in \mathcal{E}^{\uparrow}(\bar{\mathcal{V}})} - \sum_{e \in \mathcal{E}^{\downarrow}(\bar{\mathcal{V}})} (r_e - 1) > 5|\mathcal{V}|.$$

It turns out that this assumption on the graph $\mathcal{G}$ is sufficient to guarantee that the quantity $I_\lambda^{\mathcal{G}}$ has the correct scaling behavior for small λ. This is the content of the following theorem, proved in [HQ15].

Theorem 4.7. *Let $\mathcal{G}$ be a graph that satisfies Assumption 4.6, and its edges represent kernels that satisfy the definitions and bounds in* (4.7)*,* (4.8) *and* (4.9)*. If $I_\lambda^{\mathcal{G}}$ denotes the quantity defined in* (4.10)*, then one has*
$$I_\lambda^{\mathcal{G}} \lesssim \lambda^\alpha \tag{4.11}$$
uniformly over $\lambda \in (0,1)$, where $\alpha = 5|\mathcal{V} \setminus \mathcal{V}_\star| - \sum_{e \in \mathcal{E}} a_e$, and the proportionality constant depends on the graph and magnitudes of norms of the corresponding kernels.

Remark 4.8. The proportionality constant in (4.11) is a constant multiple of $\prod_e \|\widehat{J}_e\|_{a_e, p_e}$ for suitable values p_e depending on the structure of the graph, where
$$\|J\|_{a,p} := \sup_{|z| \le 1, |\ell| \le p} |z|^{a+|\ell|} |D^\ell J(z)|,$$
where we assumed that the kernels are supported in the parabolic unit ball. Since these quantities are finite, we will simply omit them in all the bounds below.

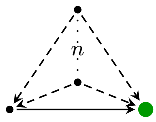
Before we prove the bounds in Proposition 4.4, we first choose values of the constants $C_n^{(\epsilon)}$ and $C_n'^{(\epsilon)}$ that appear in the statement of Theorem 4.3. With the graphic notations, the constant $C_1^{(\epsilon)}$ is given by

$$C_1^{(\epsilon)} = \int\int K_\epsilon^2(t,x) dx dt \;\; = G_\epsilon(0) =$$  $$, \qquad C_0^{(\epsilon)} = \epsilon C_1^{(\epsilon)}. \tag{4.12}$$

It is easy to see that, for this definition of $C_1^{(\epsilon)}$ and the renormalised model $\widehat{\Pi}^\epsilon$, the expression $(\widehat{\Pi}_0^\epsilon \Psi \mathcal{I}(\Psi^3))(\varphi_0^\lambda)$ only contains the first two terms in (4.5), and its variance is indeed bounded by (4.6).

For $n \geq 2$, we define $C_n^{(\epsilon)}$ and $C_n^{\prime(\epsilon)}$ by

$$C_n^{(\epsilon)} = \epsilon^{n-2} \int K(z) G_\epsilon^n(z) dz = \epsilon^{n-2}$$



$$, \quad n \geq 2 \,, \tag{4.13}$$

$$C_n^{\prime(\epsilon)} = \epsilon^{n-\frac{5}{2}} \int K(z) G_\epsilon^n(z) dz = \epsilon^{-\frac{1}{2}} C_n^{(\epsilon)}, \qquad n \geq 3 \,.$$

It is not hard to check that

$$C_1^{(\epsilon)} = \frac{C_0}{\epsilon} + \mathcal{O}(1), \quad C_0 = \int (P * \rho)^2(z) dz,$$
$$C_2^{(\epsilon)} = c_2 |\log \epsilon| + \mathcal{O}(1),$$

while for $n \geq 3$, we have

$$C_n^{(\epsilon)} = C_n + \mathcal{O}(\epsilon), \quad C_n = \int P(z)(P_\rho * P_\rho)^n(z) dz,$$

where $P_\rho = P * \rho$. C_n is finite for $n \geq 3$ since the integrand decays like $|z|^{-(n+3)}$ for large z.

4.3 First order renormalisation bounds

We are now ready to prove Theorem 4.3. In view of Proposition 4.4, it suffices to check the bound (4.2) for all terms that appear in the right hand side of (3.14), and the bounds (4.3) and (4.4) for relevant terms with positive homogeneities.

We first prove the bound (4.2) for terms from $\mathcal{F}^{(n)}$ for $n = 0, 1, 2, 3$. These basis vectors are of the form

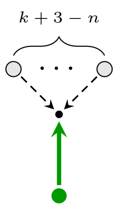
$$\tau = \mathcal{E}^{\frac{k}{2}} \Psi^{k+3-n}.$$

The case $k = 0$ has been treated in the standard Φ_3^4, so we only need to consider $k \geq 1$. For the canonical model Π^ϵ, we have

$$\Pi_z^\epsilon \tau = \epsilon^{\frac{k}{2}} (\Pi_z^\epsilon \Psi)^{k+3-n}. \tag{4.14}$$

If we choose $C_1^{(\epsilon)}$ according to (4.12), then the effect of our renormalisation procedure is precisely to turn the products in (4.14) into Wick products, so that

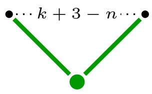
$$(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda) = \epsilon^{\frac{k}{2}}$$



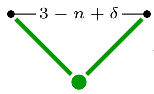
$$.$$

The right hand side belongs to the homogeneous Wiener chaos of order $(k+3-n)$, and as a consequence, we can bound its second moment by

$$\mathbf{E}|(\widehat{\Pi}_0^\epsilon\tau)(\varphi_0^\lambda)|^2 \lesssim \epsilon^k$$



$$\lesssim \epsilon^\delta$$



,

which satisfies (4.2) since

$$2|\tau| = n-3-2(k+3-n)\kappa < n-3-\delta\;,$$

if δ is small enough. The bound for $\mathcal{E}^{\frac{k}{2}}(\Psi^{k+2})X$ follows in exactly the same way. We have thus proved the bound (4.2) for $\tau = \mathcal{E}^{\frac{k}{2}}(\Psi^{k+3-n})$ and $\tau = \mathcal{E}^{\frac{k}{2}}(\Psi^{k+2})X$.

4.4 Second order renormalisation bounds

We now turn to basis vectors coming from the terms $\mathcal{F}'\mathcal{I}(\mathcal{F})$, $\mathcal{F}'\mathcal{I}(\mathcal{F}')$ and $\mathcal{F}''\mathcal{I}(\mathcal{F})$. All these basis elements have the form

$$\tau = \mathcal{E}^a(\Psi^k\mathcal{I}(\mathcal{E}^b\Psi^n)),$$

with the precise values of a and b depending on the element. For each k and n, the element

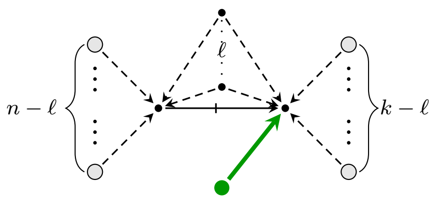
$$(\widehat{\Pi}_0^\epsilon\tau)(\varphi_0^\lambda)$$

can be decomposed into homogeneous Wiener chaoses of orders

$$k+n-2\ell, \qquad \ell = 0,1,\cdots,k\wedge n.$$

By examining the homogeneities, we notice that all the $\mathcal{E}^\beta$'s appearing in these elements play the role of multiplication by ϵ^β both under the canonical model and the Wick renormalised model. Thus, for the Wick model $\Pi_0^{\text{Wick}}(\varphi_0^\lambda)$, its component in the $(k+n-2\ell)$-th homogeneous chaos is given by

$$\ell!\binom{k}{\ell}\binom{n}{\ell}\cdot\epsilon^{a+b}$$



$$. \tag{4.15}$$

Note that the above expression is for the Wick renormalised model, and does not include effect of the map M_0 defined in Section 2.4. We now discuss the convergence for these basis elements for different values of k, n and ℓ.

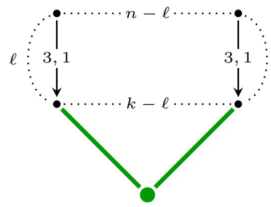
Remark 4.9. (4.15) suggests that the bounds below will in general include a labelled graph introduced above as well as a factor of a positive power of ϵ. With an abuse use of notation, in what follows, we will use $\mathcal{G}$ to denote a labelled graph multiplied a certain power of ϵ (see for example (4.16) below).

4.4.1 $k+n-2\ell \geq 2$

We show below that in this case, there is no need for renormalisation beyond Wick ordering. For simplicity, we focus on the elements from the term $\mathcal{F}'\mathcal{I}(\mathcal{F})$, and the bounds for other basis vectors follow in essentially the same way. Such basis elements have the form and homogeneity

$$\tau_{k,n} = \mathcal{E}^{\frac{k}{2}-1}(\Psi^k \mathcal{I}(\mathcal{E}^{\frac{n-3}{2}}\Psi^n)), \qquad |\tau_{k,n}| = -\frac{1}{2} - (k+n)\kappa.$$

If $k+n-2\ell \geq 2$, then as a consequence of the expression (4.15), the second moment of the component of $(\widehat{\Pi}_0^\epsilon \tau_{k,n})(\varphi_0^\lambda)$ in the $(k+n-2\ell)$-th homogeneous chaos is bounded by the graph

$$\mathcal{G} = \epsilon^{k+n-5}$$


. (4.16)

According to (4.2) and the homogeneity of $|\tau_{k,n}|$, we need to bound the graph by

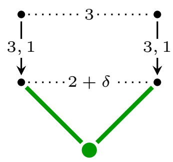
$$|I_\lambda^{\mathcal{G}}| \lesssim \epsilon^\delta \lambda^{-1-\delta} \tag{4.17}$$

for some small positive δ. The assumption that there is a positive appearance of $\mathcal{E}$ gives the condition

$$k \geq 2, \quad n \geq 3, \quad k+n \geq 6.$$

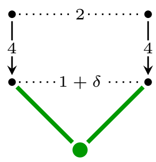
In order to get the bound (4.17), we need to assign powers of ϵ's to different edges of the graph to reduce the singularity of each edge to make the whole graph integrable. The assignments are different for various values of k, n and ℓ.

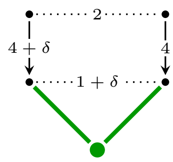
For $\ell = 0$, we can assign $(n-3)$ powers of ϵ to the upper edge and $(k-2-\delta)$ powers to the lower edge, so we obtain the bound

$$\mathcal{G} \lesssim \epsilon^\delta$$


, (4.18)

and the Assumption 4.6 can be easily checked. Thus, one gets the bound (4.17) if δ is sufficiently small. For $\ell = 1$, we still assign $(n-3)$ powers of ϵ to the upper edge and $(k-2-\delta)$ powers to the lower one, but this time the graph is reduced to

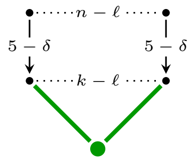
$$\mathcal{G} \lesssim \epsilon^\delta$$


$$\lesssim \epsilon^\delta$$


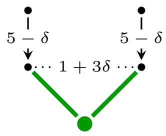
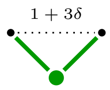
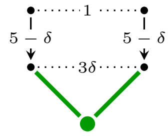
. (4.19)

Again, one can check that the conditions in Assumption 4.6 are all satisfied for this graph.

We now turn to the situation when $\ell \geq 2$. By assigning $(\ell - 2 + \delta)$ powers of ϵ to both the leftmost and the rightmost edge with weight ℓ, we reduce the graph to

$$\mathcal{G} \lesssim \epsilon^{k+n-2\ell-1-2\delta}$$  , (4.20)

and the assumption $k + n - 2\ell \geq 2$ guarantees there is still a positive power of ϵ left. If $\ell = n$, then we assign $(k - n - 1 - 3\delta)$ powers to the lower edge, and we assign $(k - n + 1 - 3\delta)$ powers to the lower edge if $\ell = n - 1$. The graphs we get in these cases becomes

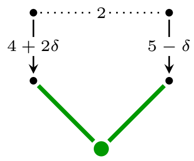
$\mathcal{G} \lesssim \epsilon^{\delta}$  $\lesssim \epsilon^{\delta}$  $(\ell = n)$, $\quad \mathcal{G} \lesssim \epsilon^{\delta}$  $(\ell = n - 1)$.

In both cases, one can easily verify Assumption 4.6 and conclude the desired bounds.

We finally turn to $n - \ell \geq 2$. In this case, we assign powers of ϵ's in the following way:

1. $(n - \ell - 2)$ powers to the upper edge;
2. $(1 - 3\delta)$ powers to the left edge;
3. and $(k - \ell)$ powers to the lower edge.

The condition $n - \ell \geq 2$ guarantees that all the powers assigned above are positive, and there is still a δ power of ϵ left. In fact, we get the reduced graph

ϵ^{δ}  $\quad (n - \ell \geq 2)$. (4.21)

Again, it is straightforward to check the Assumption (4.6) for this graph, and thus the bound (4.17) is satisfied for small enough δ. This finishes the proof of the case $k + n - 2\ell \geq 2$ for elements from $\mathcal{F}'\mathcal{I}(\mathcal{F})$. The case for the elements from the terms $\mathcal{F}'\mathcal{I}(\mathcal{F}')$ and $\mathcal{F}''\mathcal{I}(\mathcal{F})$ can be treated in essentially the same way, and we do not repeat the details here.

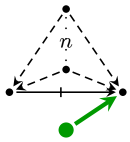
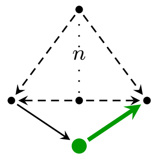
4.4.2 $k = n = \ell$

The basis elements in this category includes the following types:

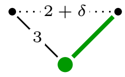
$$\mathcal{E}^{\frac{n}{2}-1}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^n)), \quad \mathcal{E}^{\frac{n-1}{2}}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n-3}{2}}\Psi^n)), \quad \mathcal{E}^{\frac{n}{2}-1}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n-3}{2}}\Psi^n)).$$

The homogeneities are just below 0 for the first two elements, and just below $-\frac{1}{2}$ for the third one. For $\ell = n$, the 0-th chaos component of the modelled distribution on these elements are just constants.

We first treat the first two elements. For both of them, the contribution to the 0-th chaos of $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ is given by

$n! \cdot \epsilon^{n-2}$  $- \; n! \cdot C_n^{(\epsilon)}$  $= \; -n! \cdot \epsilon^{n-2}$  $, \qquad (4.22)$

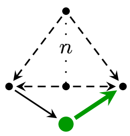
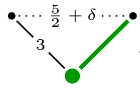
where the equality comes from the definition of the kernel $\bullet\!\!\longrightarrow\!\!\bullet$ as well as $C_n^{(\epsilon)}$ in (4.13). Since there is a strictly positive power of ϵ, by assigning $(n-2-\delta)$ powers to the dotted line in the above graph, we deduce that this object can be bounded by the graph

$\mathcal{G} = \epsilon^{\delta}$  $. \qquad (4.23)$

It is then clear that one has $I_\lambda^{\mathcal{G}} \lesssim \epsilon^{\delta} \lambda^{-\delta}$, which satisfies the bound (4.17). We now turn to the third element $\mathcal{E}^{\frac{n}{2}-1}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n-3}{2}} \Psi^n))$. The expression of the 0-th chaos is essentially the same as the previous two, except that one replaces ϵ^{n-2} by $\epsilon^{n-\frac{5}{2}}$, as well as the renormalisation constant $C_n^{(\epsilon)}$ by $C_n^{\prime(\epsilon)}$. Noting from (4.13) that

$$C_n^{\prime(\epsilon)} = \epsilon^{-\frac{1}{2}} C_n^{(\epsilon)}, \tag{4.24}$$

we obtain the expression of the 0-th chaos component of the element $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ (up to the sign) as

$n! \cdot \epsilon^{n-2}$  $\lesssim \quad \epsilon^{\delta}$  $, \qquad n \geq 3,$

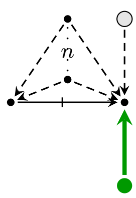
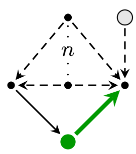
where the above bound follows from assigning $n - \frac{5}{2} - \delta$ powers of ϵ to the kernels represented by the dotted lines. This expression is bounded by $\epsilon^{\delta} \lambda^{-\frac{1}{2}-\delta}$, and corresponds to the correct homogeneity (below $-\frac{1}{2}$) if δ is sufficiently small. We have thus proved the bound (4.17) for the case $k = n = \ell$.

4.4.3 $k = n+1, \ell = n$

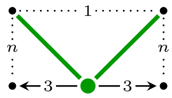
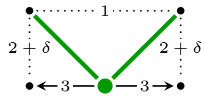
We now deal with the case $k = n+1$ and $\ell = n$, which belongs to the first order homogeneous chaos. There are two situations in this case; the first one includes basis vectors of the form

$$\tau = \mathcal{E}^{\frac{n-1}{2}}(\Psi^{n+1} \mathcal{I}(\mathcal{E}^{\frac{n-3}{2}} \Psi^n)), \qquad |\tau| = -\frac{1}{2} - (2n+1)\kappa.$$

The 1-st chaos component of $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ is given by

$(n+1)! \Bigg(\epsilon^{n-2}$  $- \; C_n^{(\epsilon)}$  $\Bigg) \; = \; -(n+1)! \cdot \epsilon^{n-2}$  $, \qquad (4.25)$

where we have used the expression of $C_n^{(\epsilon)}$ in (4.13). The second moment of this expression is then bounded (up to a constant multiple) by the graph

$$\mathcal{G} = \epsilon^{2n-4}$$  $$\lesssim \epsilon^{2\delta}$$  $$, \qquad (4.26)$$

which clearly satisfies the bound

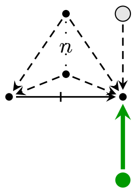
$$I_\lambda^{\mathcal{G}} \lesssim \epsilon^{2\delta} \lambda^{-1-2\delta}.$$

The exponent on λ will be bigger than twice the homogeneity of τ for small enough δ. Thus, the bound (4.17) holds for the element $\mathcal{E}^{\frac{n-1}{2}}(\Psi^{n+1}\mathcal{I}(\mathcal{E}^{\frac{n-3}{2}}\Psi^n))$.

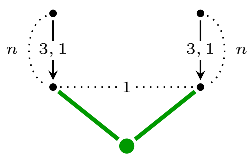
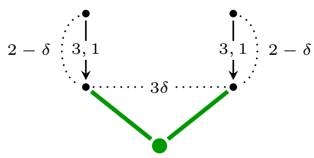
The second situation for $k = n+1$ includes the basis elements

$$\tau = \mathcal{E}^{\frac{n-1}{2}}(\Psi^{n+1}\mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^n)) \quad \text{or} \quad \tau = \mathcal{E}^{\frac{n}{2}}(\Psi^{n+1}\mathcal{I}(\mathcal{E}^{\frac{n-3}{2}}\Psi^n)).$$

In both cases, we have $|\tau| = -(2n+1)\kappa$, just below 0. Since there is no renormalisation beyond Wick ordering on these elements, the 1-st chaos component of $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ (for both of them) is given by

$$(n+1)! \cdot \epsilon^{n-\frac{3}{2}}$$  $$. \qquad (4.27)$$

The second moment of this expression is bounded by the graph

$$\mathcal{G} = \epsilon^{2n-3}$$  $$\lesssim \epsilon^{\delta}$$  $$, \qquad (4.28)$$

which immediately gives

$$I_\lambda^{\mathcal{G}} \lesssim \epsilon^{\delta} \lambda^{-3\delta}.$$

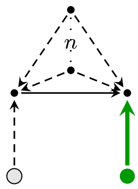
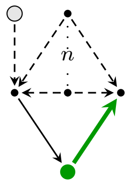
Since the homogeneities for these two τ's are below 0, we thus conclude the bound (4.17) for this case.

4.4.4 $n = k+1, \ell = k$

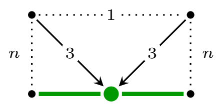
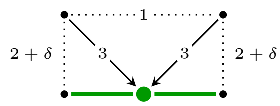
We now turn to this last case. To keep notations consistent, we switch n to $n+1$ and write the symbols as $\mathcal{E}^a(\Psi^n\mathcal{I}(\mathcal{E}^b\Psi^{n+1}))$ and $\ell = n$. The symbols in this category that need a mass renormalisation are of the form

$$\tau = \mathcal{E}^{\frac{n}{2}-1}(\Psi^n\mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^{n+1})), \quad |\tau| = -\frac{1}{2} - (2n+1)\kappa, \quad n \geq 3.$$

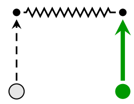
The component in the 1-st Wiener chaos of the object $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ is given by

$(n+1)!\,\Big(\epsilon^{n-2}$  $-\, C_n^{(\epsilon)}$  $\Big) \;-\; (n+1)!\cdot\epsilon^{n-2}$  . (4.29)

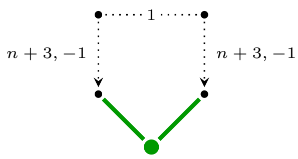
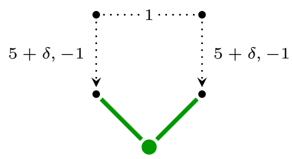
The second moment of the last term above is relatively easier to to treat. In fact, it is bounded by the graph

$\mathcal{G} = \epsilon^{n-2}$  $\lesssim\; \epsilon^{2\delta}$  , (4.30)

which clearly gives the desired bound $I_\lambda^{\mathcal{G}} \lesssim \epsilon^{2\delta}\lambda^{-1-2\delta}$. For the two terms in the parenthesis, by the definition of $C_n^{(\epsilon)}$, their difference can be expressed by the graph

ϵ^{n-2}  , (4.31)

where denotes the renormalised distribution/kernel $\mathscr{R}(KG_\epsilon^n)$, which has degree $n+3$. Thus, the second moment of this object is bounded by

ϵ^{2n-4}  $\lesssim\; \epsilon^{2\delta}$  . (4.32)

Again, one can verify that Assumption 4.6 is satisfied, and thus one has

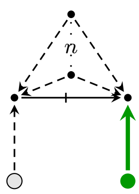
$$I_\lambda^{\mathcal{G}} \lesssim \epsilon^{2\delta}\lambda^{-1-2\delta},$$

which vanishes at the right homogeneities if δ is sufficiently small.

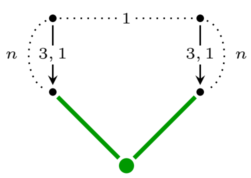
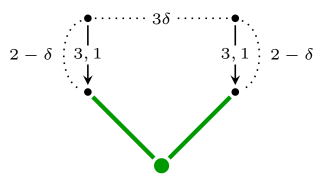
We now turn to the other two terms in this category, which are of the forms

$$\mathcal{E}^{\frac{n}{2}-1}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n-1}{2}}\Psi^n)), \qquad \mathcal{E}^{\frac{n-1}{2}}(\Psi^n \mathcal{I}(\mathcal{E}^{\frac{n}{2}-1}\Psi^n)),$$

and both have homogeneities just below 0. For both symbols, the components of $(\widehat{\Pi}_0^\epsilon \tau)(\varphi_0^\lambda)$ in the 1-st chaos can be expressed by

$(n+1)!\cdot\epsilon^{n-\frac{3}{2}}$  , (4.33)

whose second moment is bounded by the graph

$\mathcal{G} = \;\epsilon^{2n-3}$  $\lesssim\; \epsilon^{\delta}$  . (4.34)

Again, this object vanishes at the correct homogeneity. This concludes the proof of the bound (4.17) for all symbols with negative homogeneity that contains a strictly positive appearance of $\mathcal{E}$.

4.5 The bounds (4.3) and (4.4)

We first deal with the bound (4.3) on $\widehat{f}^\epsilon$. By inspection of the formal right hand side of the abstract equation, we need to prove (4.3) for $\beta = \frac{j-1}{2}$ and formal symbols τ of the form

$$\tau = \Psi^{j+2-n} \underbrace{(\mathcal{I}(\mathcal{E}^{\frac{q-1}{2}}\Psi^{q+2}))^a (\mathcal{I}(\mathcal{E}^{\frac{q-1}{2}}\Psi^{q+1}))^b X^c}_{\sigma}, \qquad n \geq 4, \quad a+b+c \leq n.$$

Since $\widehat{f}^\epsilon_z = f^{\text{Wick}}_z$, and that the Wick renormalised model $(\Pi^{\text{Wick}}, f^{\text{Wick}})$ satisfies the relation (2.11), we have

$$\widehat{f}^\epsilon_z(\mathscr{E}_0^{\frac{j-1}{2}}\tau) = -\epsilon^{\frac{j-1}{2}}(\Pi^{\text{Wick}}_z \Psi^{j+2-n})(z)(\Pi^{\text{Wick}}_z\sigma)(z),$$

where σ is the basis vector as indicated above. Since the homogeneity of σ is strictly positive, the expression above is 0 if any of the factors of σ has a positive power. Thus, the only situation we need to consider for the bound on $\widehat{f}^\epsilon_z(\mathscr{E}_0^{\frac{j-1}{2}}\tau)$ is $\tau = \Psi^{j+2-n}$, and as a consequence, we get

$$D^\ell \widehat{f}^\epsilon_z(\mathscr{E}_0^{\frac{j-1}{2}}(\Psi^{j+2-n})) = -\epsilon^{\frac{j-1}{2}}(D^\ell \Pi^{\text{Wick}}_0 \Psi^{j+2-n})(z).$$

By the definition of Π^{Wick}, the right hand side above can be expressed as a Hermite polynomial, each term being proportional to

$$\epsilon^{\frac{j-1}{2}}(C_1^{(\epsilon)})^k (D^\ell \Psi_\epsilon^{j+2-n-2k})(z) = \epsilon^{\frac{j-1}{2}}(C_1^{(\epsilon)})^k \sum_{\sum|q_i|=|\ell|} (D^{q_1}\Psi_\epsilon)(z)\cdots(D^{q_{j+2-n-2k}}\Psi_\epsilon)(z).$$

where we have written $\Psi_\epsilon = \Pi_0\Psi = K * \xi_\epsilon$ for simplicity. Now, taking expectation on the right hand side above, using generalised Hölder's inequality, and the fact that $C_1^\epsilon \sim \epsilon^{-1}$, we get

$$\mathbf{E}|D^\ell \widehat{f}^\epsilon_z(\mathscr{E}_0^{\frac{j-1}{2}}(\Psi^{j+2-n}))| \lesssim \max_{k \leq \frac{1}{2}(j+2-n)} \epsilon^{\frac{j-1}{2}-k} \sum_{\sum|q_i|=|\ell|} \prod_i \big(\mathbf{E}|D^{q_i}\Psi_\epsilon|^{j+2-n-2k}\big)^{\frac{1}{j+2-n-2k}}. \tag{4.35}$$

By equivalence of moments in Wiener chaos, each of the above factor is equivalent to $\mathbf{E}|(D^{q_i}\Psi_\epsilon)(z)|$, which could be bounded by

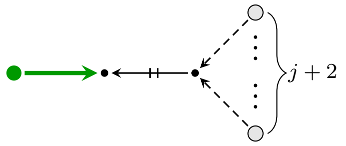
$$\mathbf{E}|(D^{q_i}\Psi_\epsilon)(z)| \lesssim (\mathbf{E}|(D^{q_i}\Psi_\epsilon)(z)|^2)^{\frac{1}{2}} \lesssim \epsilon^{-\frac{1}{2}-|q_i|}, \tag{4.36}$$

where we have used $\mathbf{E}|(D^{q_i}K * \xi_\epsilon)|^2 \lesssim \epsilon^{-1-2|q_i|}$. Combining (4.35) and (4.36), we get

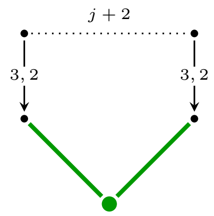
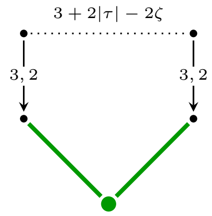
$$\mathbf{E}|D^\ell \widehat{f}^\epsilon_z(\mathscr{E}_0^{\frac{j-1}{2}}(\Psi^{j+2-n}))| \lesssim \epsilon^{\frac{j-1}{2}-\frac{j+2-n}{2}-|\ell|},$$

where we used the fact that there are totally $j+2-n-2k$ factors in the product, and $\sum |q_i| = |\ell|$. Since $|\tau| < -\frac{j+2-n}{2}$, this establishes the bound (4.3).

We now turn to the bound (4.4) for $\tau \in \mathcal{U}$, which includes Ψ, $\mathcal{I}(\mathcal{E}^{\frac{j-1}{2}}\Psi^{j+2})$, $\mathcal{I}(\mathcal{E}^{\frac{j-1}{2}}\Psi^{j+1})$, $\mathbf{1}$ and X. The bound is trivial for $\mathbf{1}$ and X, and is also straightforward for Ψ. The treatment for the rest two basis elements are similar, and we only give details for $\tau = \mathcal{I}(\mathcal{E}^{\frac{j-1}{2}}\Psi^{j+2})$. Since the test function ψ annihilates affine functions, we have

$$(\widehat{\Pi}_z^\epsilon \tau)(\psi_z^\lambda) = \epsilon^{\frac{j-1}{2}}$$  (4.37)

It then follows that we have the bound

$$\mathbf{E}|(\widehat{\Pi}_z^\epsilon \tau)(\psi_z^\lambda)|^2 \lesssim \epsilon^{j-1}$$  $$\lesssim \epsilon^{2(|\tau|-\zeta)}$$  . (4.38)

Since $2\zeta \in (2,3)$ and $2|\tau| = 1-2(j+2)\kappa$, the conditions for Assumption 4.6 can be verified straightforwardly, and thus one obtains

$$\mathbf{E}|(\widehat{\Pi}_z^\epsilon \tau)(\psi_z^\lambda)|^2 \lesssim \lambda^{1-2|\tau|+2\zeta}\epsilon^{2|\tau|-2|\zeta|} = \lambda^{2\zeta+\theta}\epsilon^{2|\tau|-2|\zeta|}$$

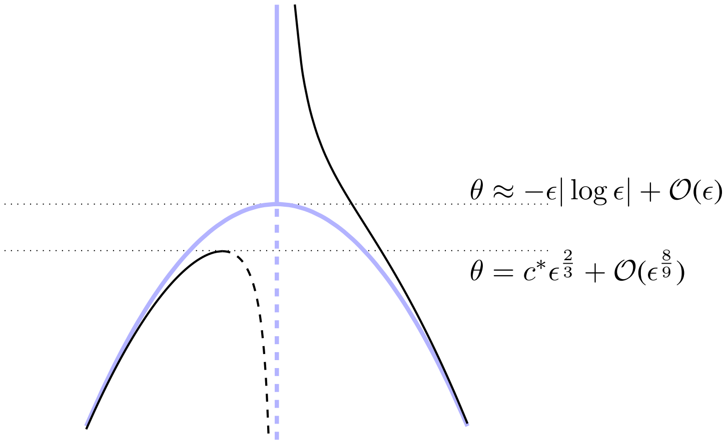
for some positive θ. The bound for $\tau = \mathcal{I}(\mathcal{E}^{\frac{j-1}{2}}\Psi^{j+1})$ follows in essentially the same way.

5 Identification of the limits

We are now ready to address the main theme of the article: identifying the large scale limits of microscopic models under various assumptions on V. As mentioned in the introduction, we will see that the large scale limit of these near-critical models are described by Φ^4_3 as long as V is symmetric, but described by either Φ^3_3 or OU processes when asymmetry is present.

The intuitive explanation of why this is so is that $\langle V \rangle$ is really only a 0-th order approximation to the "real" effective potential felt by the system at large scales. Since pitchfork bifurcations are structurally unstable, one would indeed expect higher-order corrections to $\langle V \rangle$ to turn this into a saddle-node bifurcation for generic non-symmetric potentials.

The following picture illustrates our results, with the light shaded curve representing the symmetric case and the black curve representing the generic case when $\langle V \rangle$ undergoes a pitchfork bifurcation. Here, the field Φ is represented on the horizontal axis and the bifurcation parameter θ on the vertical axis (with positive direction pointing downwards).



The reason why, in the symmetric case, we see the bifurcation at $\theta \approx -\epsilon|\log\epsilon|$ rather than $\theta \approx \epsilon$ is due to the additional mass renormalisation appearing in Φ^4_3. In the generic case where $\langle V\rangle$ is asymmetric (and the quantity A defined in (1.10) is non-zero), we can see that the asymmetry separates one local minimum from two other critical points, and creates a saddle-node bifurcation. It turns out that this bifurcation then occurs at $\theta = c^*\epsilon^{\frac{2}{3}} + \mathcal{O}(\epsilon^{\frac{8}{9}})$ for an explicitly given constant c^*.

The weak noise regime is similar to the weakly nonlinear regime, except that the critical θ at which one sees a pitchfork or saddle-node bifurcation is different. We will formulate precisely and prove these results below, starting with the weakly nonlinear regime.

5.1 Weakly nonlinear regime

Let $\tilde{u}$ be a process on a large torus satisfying

$$\partial_t \tilde{u} = \Delta\tilde{u} - \epsilon V_\theta'(\tilde{u}) + \widehat{\xi},$$

and the re-centered and rescaled process u_{ϵ^α} to be

$$u_{\epsilon^\alpha} = \epsilon^{-\frac{\alpha}{2}}(\tilde{u}(t/\epsilon^{2\alpha}, x/\epsilon^\alpha) - h),$$

where α is the scale, and h is a small parameter depending on ϵ, both to be chosen later. By setting $\delta = \epsilon^\alpha$, it is easy to see that u_δ satisfies the equation

$$\partial_t u_\delta = \Delta u_\delta - \delta^{\frac{1}{\alpha}-\frac{5}{2}} V_\theta'(\delta^{\frac{1}{2}} u_\delta + h) + \delta^{-\frac{5}{2}}\widehat{\xi}(t/\delta^2, x/\delta).$$

Note that the noise term is equivalent in law to $\xi * \rho_\delta$ for some mollifier ρ rescaled at size δ, expanding V_θ' with respect to Hermite polynomials, we get

$$\partial_t u_\delta = \Delta u_\delta - \delta^{\frac{1}{\alpha}-1} \sum_{j=0}^{m} \widehat{a}_j^{(h)}(\theta) \cdot \delta^{\frac{j-3}{2}} H_j(u_\delta; C_1^{(\delta)}) + \xi_\delta, \tag{5.1}$$

where

$$\widehat{a}_j^{(h)}(\theta) = \sum_{k=j}^{m} \binom{k}{j} \widehat{a}_k(\theta) \cdot h^{k-j}.$$

We now fix $\gamma \in (1, \frac{6}{5})$, $\eta \in (-\frac{m+1}{2m}, \frac{1}{2})$, and we shall lift the above equation to the abstract $\mathcal{D}_\epsilon^{\gamma,\eta}$ space associated to the model $\mathfrak{M}_\delta = M_\delta \mathscr{L}_\delta(\xi_\delta)$ as in Theorem 4.3. We also let $\phi_0^{(\delta)} \in \mathcal{C}_\delta^{\gamma,\eta}$ such that $\|\phi_0^{(\delta)}; \phi_0\|_{\gamma,\eta;\delta} \to 0$ for some $\phi_0 \in \mathcal{C}^\eta$. The corresponding abstract fixed point equation then has the form

$$\Phi^{(\delta)} = \mathcal{P}\mathbf{1}_+ \Big(\Xi - \sum_{j=4}^{m} \lambda_j^{(\delta)} \mathcal{Q}_{\leq 0} \widehat{\mathcal{E}}^{\frac{j-3}{2}} \mathcal{Q}_{\leq 0}((\Phi^{(\delta)})^j) - \sum_{j=0}^{3} \lambda_j^{(\delta)} \mathcal{Q}_{\leq 0}((\Phi^{(\delta)})^j) \Big) + \widehat{P}\phi_0^{(\delta)}. \tag{5.2}$$

Comparing the right hand sides of (3.15) and (5.1), we should choose the coefficients $\lambda_j^{(\delta)}$'s to be

$$\begin{aligned}
\lambda_j^{(\delta)} &= \delta^{\frac{1}{\alpha}-1} \cdot \widehat{a}_j^{(h)}(\theta), \qquad j \geq 3; \\
\lambda_2^{(\delta)} &= \delta^{\frac{1}{\alpha}-\frac{3}{2}} \cdot \widehat{a}_2^{(h)}(\theta); \\
\lambda_1^{(\delta)} &= \delta^{\frac{1}{\alpha}-2} \cdot \widehat{a}_1^{(h)}(\theta) - \delta^{\frac{2}{\alpha}-2} C_{\delta,\theta,h}; \\
\lambda_0^{(\delta)} &= \delta^{\frac{1}{\alpha}-\frac{5}{2}} \cdot \widehat{a}_0^{(h)}(\theta) - \delta^{\frac{2}{\alpha}-\frac{5}{2}} C'_{\delta,\theta,h} - 6\lambda_2^{(\delta)}\lambda_3^{(\delta)} C_2^{(\delta)},
\end{aligned}$$

where

$$\begin{aligned}
C_{\delta,\theta,h} &= \sum_{n=2}^{m-1} (n+1)^2 n! \cdot (\widehat{a}_{n+1}^{(h)}(\theta))^2 \cdot C_n^{(\delta)} + \sum_{n=3}^{m-2} (n+2)! \cdot \widehat{a}_n^{(h)}(\theta) \cdot \widehat{a}_{n+2}^{(h)}(\theta) \cdot C_n^{(\delta)} \\
&= 18\widehat{a}_3^2 c_2 |\log\delta| + \mathcal{O}(1); \\
C'_{\delta,\theta,h} &= \sum_{n=3}^{m-1} (n+1)! \cdot \widehat{a}_n^{(h)}(\theta) \cdot \widehat{a}_{n+1}(\theta) C_n^{(\delta)} = A + \mathcal{O}(\delta,\theta,h),
\end{aligned}$$

Here, the quantity A is given by

$$A = \sum_{n=3}^{m-1} (n+1)! \cdot \widehat{a}_n \widehat{a}_{n+1} C_n, \tag{5.3}$$

and C_n's are the limits of $C_n^{(\delta)}$'s (recall that they do converge to a finite limit for $n \geq 3$). It is then clear that the reconstructed solution $u_\delta = \widehat{\mathcal{R}}^\delta \Phi^{(\delta)}$ exactly solves (5.1) with initial condition $\phi_0^{(\delta)}$. Here, we have used the notation $\mathcal{O}(a,b)$ to denote $\mathcal{O}(a \vee b)$.

By Theorem 4.3, there exists a limiting model $\mathfrak{M} \in \mathscr{M}_0$ such that $\|\mathfrak{M}_\delta; \mathfrak{M}\|_{\delta;0} \to 0$. If $\lambda_j^{(\delta)}$ converges to some $\lambda_j \in \mathbf{R}$ for each j, then by Theorem 3.12, we will have $\|\Phi^{(\delta)}; \Phi\|_{\gamma,\eta;\delta} \to 0$, where $\Phi \in \mathcal{D}^{\gamma,\eta}$ associated to the model $\mathfrak{M}$ solves the fixed point equation

$$\Phi = \mathcal{P}\mathbf{1}_+ \Big(\Xi - \sum_{j=4}^{m} \lambda_j \mathcal{Q}_{\leq 0} \widehat{\mathcal{E}}^{\frac{j-3}{2}} (\mathcal{Q}_{\leq 0}(\Phi^j)) - \sum_{j=0}^{3} \lambda_j \mathcal{Q}_{\leq 0}(\Phi^j) \Big) + \widehat{P}\phi_0. \tag{5.4}$$

The continuity of the reconstruction operator thus implies $u_\delta \to u = \widehat{\mathcal{R}}\Phi$ in $\mathcal{C}^\eta$. In what follows, we will choose the small parameter h as well as the scale α in a proper way such that the coefficients $\lambda_j^{(\delta)}$'s do converge to the desired limiting values under various assumptions on V. Once these limiting values λ_j's are known, we can immediately derive the limiting equation that u solves. We will also always assume that u_δ solves (5.1) on $[0,T]\times \mathbf{T}^3$ with initial condition $\phi_0^{(\delta)}$.

We now assume that $\langle V_\theta \rangle$ satisfies a pitchfork bifurcation at $(0,0)$. Then, by the conditions (1.8) and (1.9), the coefficients $\widehat{a}_j^{(h)}(\theta)$ on the right hand side of (5.1) satisfy

$$\begin{aligned}
&\widehat{a}_j^{(h)}(\theta) = \widehat{a}_j + \mathcal{O}(\theta, h), \qquad j \geq 3;\\
&\widehat{a}_2^{(h)}(\theta) = 3\widehat{a}_3 h + \mathcal{O}(\theta, h^2);\\
&\widehat{a}_1^{(h)}(\theta) = 3\widehat{a}_3 h^2 + \widehat{a}_1'\theta + \mathcal{O}(\theta^2, \theta h, h^3);\\
&\widehat{a}_0^{(h)}(\theta) = \widehat{a}_3 h^3 + \widehat{a}_1'\theta h + \frac{\widehat{a}_0''}{2}\cdot \theta^2 + \mathcal{O}(\theta^3, \theta h^2, h^4),
\end{aligned} \tag{5.5}$$

As already mentioned in the introduction, whether one could obtain Φ_3^4 in the large scale limit depends on whether the quantity A defined in (5.3) is 0. In the case $A=0$, we have the following theorem.

Theorem 5.1. *Let $A=0$. If we set $\alpha=1$, $h=0$, and*

$$\theta = \theta(\epsilon) = \frac{18\widehat{a}_3^2 c_2}{\widehat{a}_1'}\cdot \epsilon|\log\epsilon| + \lambda\epsilon + o(\epsilon),$$

then u_ϵ converges in probability in $\mathcal{C}^\eta([0,T]\times \mathbf{T}^3)$ to the $\Phi_3^4(\widehat{a}_3)$ family of solutions indexed by λ with initial condition ϕ.

Proof. Since $\alpha=1$, we actually have $\epsilon=\delta$. From (5.5), we immediately deduce that

$$\lambda_j^{(\epsilon)} \to \widehat{a}_j, \qquad j\geq 3.$$

Since $h=0$, we have $\widehat{a}_2(\theta)\sim \epsilon\log\epsilon$, which gives $\lambda_2^{(\epsilon)}\sim \epsilon^{\frac12}\log\epsilon \to 0$. For $\lambda_0^{(\epsilon)}$, we have

$$\widehat{a}_0(\theta)\sim \epsilon^2\log^2\epsilon, \qquad \lambda_2^{(\epsilon)} = \mathcal{O}(\epsilon^{\frac12}\log\epsilon),$$

so the only problematic term is $C_{\epsilon,\theta}'$. But note that $A=0$, this term also vanishes, so we also have $\lambda_0^{(\epsilon)}\to 0$.

We now turn to $\lambda_1^{(\epsilon)}$. Note that both $\widehat{a}_1(\theta)\cdot\epsilon^{-1}$ and $C_{\epsilon,\theta}$ diverge logarithmically, but the prefactor of the term $\epsilon|\log\epsilon|$ in θ guarantees that these two divergent terms cancel each other, so $\lambda_1^{(\epsilon)}$ converges to some finite quantity λ_1, depending on the choice λ in front of the ϵ term in θ. This implies that when restricted to basis vectors without an appearance of $\mathcal{E}$, the formal right hand side of (5.4) is identical as that of $\Phi_3^4(\widehat{a}_3)$ with a proper linear term.

Since the action of the model $\mathfrak{M}$ on basis vectors without an appearance of $\mathcal{E}$ are precisely the same as the limiting model in Φ_3^4, and its action on symbols with $\mathcal{E}$ yields 0, it then follows immediately that $u=\widehat{\mathcal{R}}\Phi$ for the limiting equation does coincide with the $\Phi_3^4(\widehat{a}_3)$ family. This completes the proof. $\square$

Note that the leading order term of $\theta(\epsilon)$ is negative since $\widehat{a}_1' < 0$. We now turn to the asymmetric case where $A \neq 0$. We could assume without loss of generality that $A > 0$. The main statement is the following.

Theorem 5.2. *Let $A > 0$, and assume $\theta = \rho\epsilon^\beta$ near the origin ($\rho > 0$).*

1. *If $\beta < \frac{2}{3}$, then there exists three distinct choices $h_\epsilon^{(1)} < h_\epsilon^{(2)} < h_\epsilon^{(3)}$ such that at scale $\alpha = \frac{1+\beta}{2}$, both $u_\delta^{(1)}$ and $u_\delta^{(3)}$ converges in probability to u while $u_\delta^{(2)}$ converges in probability to v, where u and v solves the equations*

$$\partial_t u = \Delta u - 2|\widehat{a}_1'|\rho u + \xi, \qquad \partial_t v = \Delta v + |\widehat{a}_1'|\rho v + \xi,$$

respectively, both with initial data ϕ_0.

2. *If $\beta > \frac{2}{3}$, then there exists a unique choice h_ϵ such that at scale $\alpha = \frac{5}{6}$, the process u_δ converges in probability to the solution u of the equation*

$$\partial_t u = \Delta u - 3\left(\widehat{a}_3 A^2\right)^{\frac{1}{3}} u + \xi$$

with initial data ϕ_0.

3. *If $\beta = \frac{2}{3}$ and $\theta = \rho\epsilon^{\frac{2}{3}}$, then there exists a critical value*

$$\rho^* = \frac{3}{|\widehat{a}_1'|} \cdot \left(\frac{\widehat{a}_3 A^2}{4}\right)^{\frac{1}{3}} \tag{5.6}$$

such that for $\rho < \rho^*$ (and resp. $\rho > \rho^*$) there exist one (and three, resp.) choices of h such that at scale $\alpha = \frac{5}{6}$, u_{ϵ^α} converges to one or three distinct O.U. processes.

For $\rho = \rho^*$, there exist two distinct choices $h_\epsilon^{(1)} < h_\epsilon^{(2)}$ such that for $h = h_\epsilon^{(1)}$, at scale $\alpha = \frac{8}{9}$, u_{ϵ^α} converges to the solution u of the equation

$$\partial_t u = \Delta u + 3\left(\frac{\widehat{a}_3^2 A}{2}\right)^{\frac{1}{3}} :u^2: \; + \xi,$$

while for $h = h_\epsilon^{(2)}$, u_{ϵ^α} still converges to O.U. at scale $\alpha = \frac{5}{6}$.

All the convergences above are in $\mathcal{C}^\eta([0,T] \times \mathbf{T}^3)$.

Remark 5.3. The situation for $\beta = \frac{2}{3}$ and $\rho < \rho^*$ (or $\rho > \rho^*$) are similar to that of $\beta > \frac{2}{3}$ (or $\beta < \frac{2}{3}$), except that the coefficients in the limiting equations are different. The other difference is that in the case $\rho > \rho^*$, the three choices of h gives three different limiting O.U. processes, unlike when $\beta < \frac{2}{3}$, two of the three h's gives the same limiting equation.

We will give the proof of the above theorem for the most interesting case $\beta = \frac{2}{3}$, and the proof for the other two situations are essentially the same but only simpler. We will make use of the following elementary lemma.

Lemma 5.4. *Let $A > 0$. For any $\rho > 0$, let $f_\rho(r) = \widehat{a}_3 r^3 + \rho \widehat{a}_1' r - A$. Let ρ^* be the same as in* (5.6). *Then, the equation $f_\rho(r) = 0$ has one, two, or three distinct real roots for $\rho < \rho^*$, $\rho = \rho^*$ and $\rho > \rho^*$, respectively. In particular, for $\rho = \rho^*$, the two roots $r_1 < r_2$ satisfy*

$$r_1 = -\left(\frac{A}{2\widehat{a}_3}\right)^{\frac{1}{3}}, \qquad r_2 > \left(\frac{A}{2\widehat{a}_3}\right)^{\frac{1}{3}}.$$

Proof. If $f_\rho(r) = 0$ has exactly two distinct roots, then since $A > 0$, the smaller one must also be a local maximum for f_ρ. The value of ρ^* and that root could then be computed directly, and all other assertions follow. □

Proof of Theorem 5.2.

We only give details to the case when $\beta = \frac{2}{3}$ so $\theta = \rho\epsilon^{\frac{2}{3}}$. For $\rho = \rho^*$, let $r_1 < r_2$ be the two roots to the equation $f_{\rho^*}(r) = 0$, and set

$$\begin{aligned} \alpha_1 &= \frac{8}{9}, \qquad \theta \sim \rho^* \delta_1^{\frac{3}{4}}, \qquad h_\delta^{(1)} = r_1 \delta_1^{\frac{3}{8}} = r_1 \epsilon^{\frac{1}{3}}; \\ \alpha_2 &= \frac{5}{6}, \qquad \theta \sim \rho^* \delta_2^{\frac{4}{5}}, \qquad h_\delta^{(2)} = r_2 \delta_1^{\frac{2}{5}} = r_2 \epsilon^{\frac{1}{3}}, \end{aligned}$$

For the choice of $(\alpha_1, h^{(1)})$, we deduce from the properties of $\widehat{a}_j^{(h)}(\theta)$'s that $\lambda_j^{(\delta_1)} \to 0$ for all $j \neq 2$, while

$$\lambda_2^{(\delta_1)} \to -3\left(\frac{\widehat{a}_3^2 A}{2}\right)^{\frac{1}{3}}.$$

The claim then follows immediately. For the choice $(\alpha_2, h^{(2)})$, we have $\lambda_j^{(\delta_2)} \to 0$ for all $j \neq 1$ and $\lambda_1^{(\delta_2)}$ converges to some positive real number. Thus, the limiting process in this case is O.U..

For $\rho < \rho^*$ and $\rho > \rho^*$, one should note that there exist one (or three, respectively) distinct real solutions to the equation

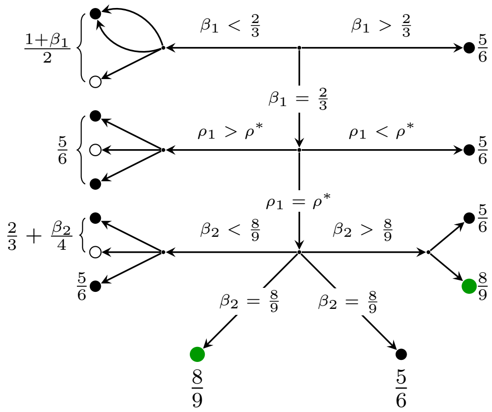
$$f_\rho(r) = 0.$$

By setting $\alpha = \frac{5}{6}$ and $h_\delta = r\delta^{\frac{2}{5}} = r\epsilon^{\frac{1}{3}}$ with the roots r, one can show that all $\lambda_j^{(\delta)}$'s vanish in the limit except $\lambda_1^{(\delta)}$ which converges to a finite quantity. The form of the limiting equation then follows immediately. The coefficient of the drift term can be found by computing the roots to $f_\rho(r) = 0$, but this is not important here. This completes the proof.

Remark 5.5. One can also adjust θ to the second order. In fact, for

$$\theta = \rho_1 \epsilon^{\beta_1} + \rho_2 \epsilon^{\beta_2}$$

with $\beta_1 = \frac{2}{3}$ and $\rho_1 = \rho^*$, it is not difficult to show that if $\beta_2 < \frac{8}{9}$ and $\rho_2 > 0$, then one still gets three OU's, but two of them are observed at larger scales than $\frac{5}{6}$. If $\beta_2 \geq \frac{8}{9}$, then one can get Φ^3_3. This can be illustrated by the following figure.



Here, each $\bullet$ represents a stable OU process (the one with two arrows pointing to it indicates that the two limiting OU processes have the same coefficient), each $\circ$ represents an unstable OU process, and the green node $\bullet$ represents a Φ^3_3 equation. The difference between the two green dots are that the limit represented by the one at the bottom represents a Φ^3_3 family parametrised by the coefficient ρ_2, while the one on the right has the canonical Wick product meaning. Finally, the numbers next to each node indicates the scale α.

Note that for the branch containing the saddle-node bifurcation, the scale increases from $\frac{1}{2}$ to $\frac{8}{9}$ continuously with respect to the exponents (β_1, β_2). One can also obtain such a complete description for the symmetric case, but we omit the statement of the details for conciseness. Also, the reason that the three nodes $\bullet$ on the right of the figure all exhibit scale $\frac{5}{6}$ is that we only include the case when $\rho_1 > 0$. In fact, one can also recover the scales from $\frac{1}{2}$ to $\frac{5}{6}$ continuously by considering $\beta_1 < \frac{2}{3}$ and $\rho_1 < 0$.

Remark 5.6. We now very briefly discuss the case when $\langle V_\theta \rangle$ has a stable extreme point or a saddle-node bifurcation near the origin. The proofs are much simpler than the pitchfork bifurcation case, so we do not give details here. In both cases, no re-centering is needed so $h = 0$.

If $\langle V \rangle$ has a stable extreme point at the origin, then $\widehat{a}_1 \neq 0$. In this case, we choose $\alpha = \frac{1}{2}$ (so $\delta = \epsilon^{\frac{1}{2}}$). Since we always assume $\widehat{a}_0 = 0$, then as long as $\theta = o(\epsilon)$, all $\lambda^{(\delta)}_j$'s vanish in the limit except $\lambda^{(\delta)}_1 \to \widehat{a}_1$. Thus, the process u_δ converges in probability to the limit

$$\partial_t u = \Delta u - \widehat{a}_1 u + \xi.$$

In the case of saddle-node bifurcation when $\widehat{a}_0 = \widehat{a}_1 = 0$ but $\widehat{a}_2 \neq 0$, the correct scale here should be $\alpha = \frac{2}{3}$. Then, as long as $\theta = o(\delta) = o(\epsilon^{\frac{3}{2}})$, all $\lambda^{(\delta)}_j \to 0$ except for $\lambda^{(\delta)}_2$ which converges to $\widehat{a}_2$. This gives the limiting equation

$$\partial_t u = \Delta u - \widehat{a}_2 :u^2: + \xi.$$

If $\theta = \mathcal{O}(\epsilon^{\frac{3}{2}})$, then the resulting limit is a Φ^3_3 family. Note that in the above two cases, no further renormalisation is needed beyond the usual Wick ordering, so they can actually be treated using the methods developed in [DPD03] and [EJS13].

5.2 Weak noise regime

We now consider the weak noise regime. Here, we assume $V : \theta \mapsto V_\theta(\cdot)$ is smooth in $\mathcal{C}^8$ functions so that it can be expanded near $x = 0$ as in (1.15). We also assume that V has a pitchfork bifurcation near the origin in the sense of (1.16). Let $\tilde{u}$ be the process satisfying

$$\partial_t \tilde{u} = \Delta \tilde{u} - V'_\theta(\tilde{u}) + \epsilon^{\frac{1}{2}} \widehat{\xi},$$

and define u_{ϵ^α} to be

$$u_{\epsilon^\alpha} = \epsilon^{-\frac{1+\alpha}{2}} (\tilde{u}(t/\epsilon^{2\alpha}, x/\epsilon^\alpha) - h).$$

By setting $\delta = \epsilon^\alpha$, we see that u_δ satisfies the equation

$$\partial_t u_\delta = \Delta u_\delta - \sum_{j=0}^{6} a_j^{(h)}(\theta) \delta^{\frac{j-1}{2\alpha} + \frac{j-5}{2}} u_\delta^j - \delta^{-\frac{1}{2\alpha} - \frac{5}{2}} F_{\theta,h}(\delta^{\frac{1}{2\alpha} + \frac{1}{2}} u_\delta) + \xi_\delta \tag{5.7}$$

for certain function $F_{\theta,h}$ satisfying $|F_{\theta,h}(x)| \lesssim |x|^7$ uniformly over $|\theta|, |h|, |x| < 1$, and the coefficients $a_j^{(h)}$'s satisfy

$$a_j^{(h)}(\theta) = \sum_{k=j}^{6} a_k(\theta) \binom{k}{j} \cdot h^{k-j} + \mathcal{O}(h^{7-j}), \qquad 0 \le j \le 6. \tag{5.8}$$

Similar as before, we always assume (5.7) starts with initial data $\phi_0^{(\delta)} \in \mathcal{C}_\delta^{\gamma,\eta}$ such that $\|\phi_0^{(\delta)}; \phi_0\|_{\gamma,\eta;\delta} \to 0$ for some $\phi_0 \in \mathcal{C}^\eta$.

We still let $\mathfrak{M}_\delta = M_\delta \mathscr{L}_\delta(\xi_\delta)$ be the renormalised model as before, $\mathcal{D}_\delta^{\gamma,\eta}$ and $\widehat{\mathcal{R}}^\delta$ be the associated space and reconstruction operator, and consider the abstract fixed point equation

$$\begin{aligned}\Phi^{(\delta)} =& \mathcal{P}\mathbf{1}_+ \bigg(\Xi - \sum_{j=4}^{6} \lambda_j^{(\delta)} \mathcal{Q}_{\le 0} \widehat{\mathcal{E}}^{\frac{j-3}{2}} \mathcal{Q}_{\le 0}((\Phi^{(\delta)})^j) - \sum_{j=0}^{3} \lambda_j^{(\delta)} \mathcal{Q}_{\le 0}((\Phi^{(\delta)})^j) \\ & - \delta^{-\frac{1}{2\alpha} - \frac{5}{2}} F_{\theta,h}(\delta^{\frac{1}{2\alpha} + \frac{1}{2}} \widehat{\mathcal{R}}^\delta \Phi^\delta) \cdot \mathbf{1} \bigg) + \widehat{P} \phi_0^{(\delta)}.\end{aligned} \tag{5.9}$$

Here, we allow the parameters θ and h to depend on δ, which is indeed the case we consider later. The following statement is an analogy to Theorem 3.12. It will be crucial to proving the convergence of u_δ to corresponding limits in various situations.

Theorem 5.7. *Let $\mathfrak{M}_\delta \in \mathscr{M}_\delta$ and $\mathfrak{M} \in \mathscr{M}$ be as before, and let $\alpha \le 1$. Suppose $|F_{\theta,h}(x)| \lesssim |x|^7$ near the origin uniformly over $|\theta|, |h| < 1$, and suppose for each j, there exists $\lambda_j \in \mathbf{R}$ such that $\lambda_j^{(\delta)} \to \lambda_j$. Then, there exists a short existence time T such that there is a unique fixed point solution $\Phi \in \mathcal{D}^{\gamma,\eta}$ to the equation*

$$\Phi = \mathcal{P}\mathbf{1}_+ \bigg(\Xi - \sum_{j=4}^{6} \lambda_j \mathcal{Q}_{\le 0} \widehat{\mathcal{E}}^{\frac{j-3}{2}} (\mathcal{Q}_{\le 0}(\Phi^j)) - \sum_{j=0}^{3} \lambda_j \mathcal{Q}_{\le 0}(\Phi^j) \bigg) + \widehat{P} \phi_0.$$

Furthermore, for every small enough δ, there also exists a fixed point solution $\Phi^{(\delta)} \in \mathcal{D}_\delta^{\gamma,\eta}$ to (5.9) up to the same time T such that $\|\!|\Phi^{(\delta)}, \Phi|\!\|_{\gamma,\eta;\delta,0} \to 0$.

Proof. In view of Theorem 3.12, it suffices to prove that the map (up to some fixed time S)

$$\Phi^{(\delta)} \mapsto \delta^{-\frac{1}{2\alpha}-\frac{5}{2}} \mathcal{P}\mathbf{1}_+ \big(F_{\theta,h}(\delta^{\frac{1}{2\alpha}+\frac{1}{2}} \widehat{\mathcal{R}}^\delta \Phi^\delta) \cdot \mathbf{1}\big) \tag{5.10}$$

is locally Lipschitz from $\mathcal{D}_\delta^{\gamma,\eta}$ to itself with a Lipschitz constant bounded by δ^σ for some positive σ, uniformly over θ and h. We need this uniformity because of the dependence of θ and h on δ in (5.9).

To see (5.10), we first note that if Φ solves the fixed point equation (5.9), then it necessarily has the form

$$\Phi = \Psi + U(z),$$

where U takes value in a subspace of $\mathcal{T}$ spanned by $\mathbf{1}$ and elements with strictly positive homogeneities. As a consequence, we have

$$(\widehat{\mathcal{R}}^\delta U)(z) = \langle U(z), \mathbf{1} \rangle \lesssim (\delta + \sqrt{|t|})^\eta \|U\|_{\gamma,\eta;\delta}.$$

It is also straightforward to show that

$$|(\widehat{\mathcal{R}}\Psi)(z)| = |(K * \xi_\delta)(z)| \lesssim \delta^{-\frac{1}{2}-\kappa} \|\mathfrak{M}_\delta\|_\delta.$$

Thus, combining the above two bounds together with the assumption of the behavior of F around 0, we deduce that the map

$$\Phi^{(\delta)} \mapsto \delta^{-\frac{1}{2\alpha}-\frac{5}{2}} F_{\theta,h}(\delta^{\frac{1}{2\alpha}+\frac{1}{2}} \widehat{\mathcal{R}}^\delta \Phi^\delta) \cdot \mathbf{1}$$

is locally Lipschitz continuous from $\mathcal{D}_\delta^{\gamma,\eta}$ to the space of continuous functions $\mathcal{C}$ with uniform topology, and that the local Lipschitz constant is proportional to δ^σ for some $\sigma > 0$ (independent of θ and h). The additional operation by $\mathcal{P}\mathbf{1}_+$ (up to time S) makes the map (5.10) locally Lipschitz from $\mathcal{D}_\delta^{\gamma,\eta}$ to itself, and the Lipschitz constant is bounded by $(S\delta)^\sigma$.

The rest of the proof follows in the same line as that in Theorem 3.12. □

Suppose we have now chosen $\lambda_j^{(\delta)}$'s such that $\widehat{\mathcal{R}}^\delta \Phi^{(\delta)}$ exactly solves (5.7). By the assumptions on the models and initial conditions, Theorem 5.7 guarantees that as long as we can show that these $\lambda_j^{(\delta)}$'s converge to the desired limiting values, then the convergence of u_δ to the limiting process with follow automatically as in the previous section.

Inspecting the right hand side of (3.15), we see that in order for $\widehat{\mathcal{R}}^\delta \Phi^{(\delta)}$ to solve (5.7), we need to set $\lambda_j^{(\delta)}$'s in the following way:

$$\begin{aligned}
\lambda_6^{(\delta)} &= a_6^{(h)}(\theta) \cdot \delta^{\frac{5}{2\alpha}-1}, \qquad \lambda_5^{(\delta)} = a_5^{(h)}(\theta) \cdot \delta^{\frac{2}{\alpha}-1};\\
\lambda_4^{(\delta)} &= \delta^{\frac{3}{2\alpha}-1}\big(a_4^{(h)}(\theta) + 15 a_6^{(h)}(\theta) C_0 \cdot \delta^{\frac{1}{\alpha}}\big);\\
\lambda_3^{(\delta)} &= \delta^{\frac{1}{\alpha}-1}\big(a_3^{(h)}(\theta) + 10 a_5^{(h)}(\theta) C_0 \cdot \delta^{\frac{1}{\alpha}}\big);\\
\lambda_2^{(\delta)} &= \delta^{\frac{1}{2\alpha}-\frac{3}{2}}\big(a_2^{(h)}(\theta) + 6 a_4^{(h)}(\theta) C_0 \cdot \delta^{\frac{1}{\alpha}} + 45 a_6^{(h)}(\theta) C_0^2 \cdot \delta^{\frac{2}{\alpha}}\big);\\
\lambda_1^{(\delta)} &= \delta^{-2}\big(a_1^{(h)}(\theta) + 3 a_3^{(h)}(\theta) C_0 \cdot \delta^{\frac{1}{\alpha}} + 15 a_5^{(h)}(\theta) C_0^2 \cdot \delta^{\frac{2}{\alpha}}\big) - C_\delta;\\
\lambda_0^{(\delta)} &= \delta^{-\frac{1}{2\alpha}-\frac{5}{2}}\big(a_0^{(h)}(\theta) + a_2^{(h)}(\theta) C_0 \cdot \delta^{\frac{1}{\alpha}} + 3 a_4^{(h)}(\theta) C_0^2 \cdot \delta^{\frac{2}{\alpha}} + 15 a_6^{(h)}(\theta) C_0^3 \cdot \delta^{\frac{3}{\alpha}}\big)\\
&\quad - C_\delta' - 6 \lambda_2^{(\delta)} \lambda_3^{(\delta)} C_2^{(\delta)},
\end{aligned} \tag{5.11}$$

where the constants C_δ and C'_δ are given by

$$C_\delta = \sum_{n=2}^{5}(n+1)^2 n! \cdot (\lambda_{n+1}^{(\delta)})^2 C_n^{(\delta)} + \sum_{n=3}^{4}(n+2)! \cdot \lambda_n^{(\delta)}\lambda_{n+2}^{(\delta)} C_n^{(\delta)};$$

$$C'_\delta = \delta^{-\frac{1}{2}} \sum_{n=3}^{5}(n+1)! \cdot \lambda_n^{(\delta)}\lambda_{n+1}^{(\delta)} C_n^{(\delta)}$$

The additional term $F_{\theta,h} \cdot \mathbf{1}$ in (5.9) does not affect the choice as it precisely gives the corresponding term in (5.7) when hit with the reconstruction operator. The following statement gives the situation where we can observe Φ_3^4.

Theorem 5.8. *Suppose*

$$B = a_4 + \frac{3a''_0 a_3^2}{2a_1'^2} - \frac{a'_2 a_3}{a'_1} = 0. \tag{5.12}$$

Then, there exists

$$\theta(\epsilon) = -\frac{3a_3 C_0}{a'_1} \cdot \epsilon + \frac{18a_3^2 c_2}{a'_1} \cdot \epsilon^2 |\log \epsilon| + \lambda \epsilon^2, \qquad h(\epsilon) = \rho\epsilon + \mathcal{O}(\epsilon^2),$$

such that at scale $\alpha = 1$, the solution u_ϵ to (5.7) *with initial condition $\phi_0^{(\epsilon)}$ converges in probability in $\mathcal{C}^\eta([0,T] \times \mathbf{T}^3)$ to the $\Phi_3^4(a_3)$ family (with initial data ϕ_0) with an additional constant.*

Proof. At $\alpha = 1$, we have $\delta = \epsilon$. It is easy to see that if $B = 0$, then with the above choice of θ, all $\lambda_j^{(\epsilon)}$'s converge to a finite limit. In particular, we have

$$\lambda_j^{(\epsilon)} \to 0 \ \ (j \geq 4), \qquad \lambda_3^{(\epsilon)} \to a_3, \qquad \lambda_2^{(\epsilon)} \to \lambda_2 = -\frac{3a'_2 a_3 C_0}{a'_1} + 3a_3\rho + 6a_4 C_0.$$

Since $a_3 \neq 0$, we can choose ρ such that $\lambda_2 = 0$. For $\lambda_0^{(\epsilon)}$, it is straightforward to show that it converges to a finite limiting λ_0 whose value depends on λ. The assertion then follows from Theorem 5.7 and the continuity of the reconstruction operators. □

Remark 5.9. It is clear from the proof that the role of h is to kill the quadratic Wick term on the right hand side of the limiting equation. One could also set $h = 0$, but then the limiting equation will involve both a quadratic Wick term and a constant.

In the generic case when $B \neq 0$, we need to look at a different scale to observe a non-trivial limit. The critical value of θ at which one sees a saddle-node bifurcation turns out to be

$$\theta^*(\epsilon) = \rho_1^*\epsilon + \rho_2^*\epsilon^{\frac{4}{3}} + \rho_3^*\epsilon^{\frac{5}{3}} + \mathcal{O}(\epsilon^{\frac{16}{9}})$$

with

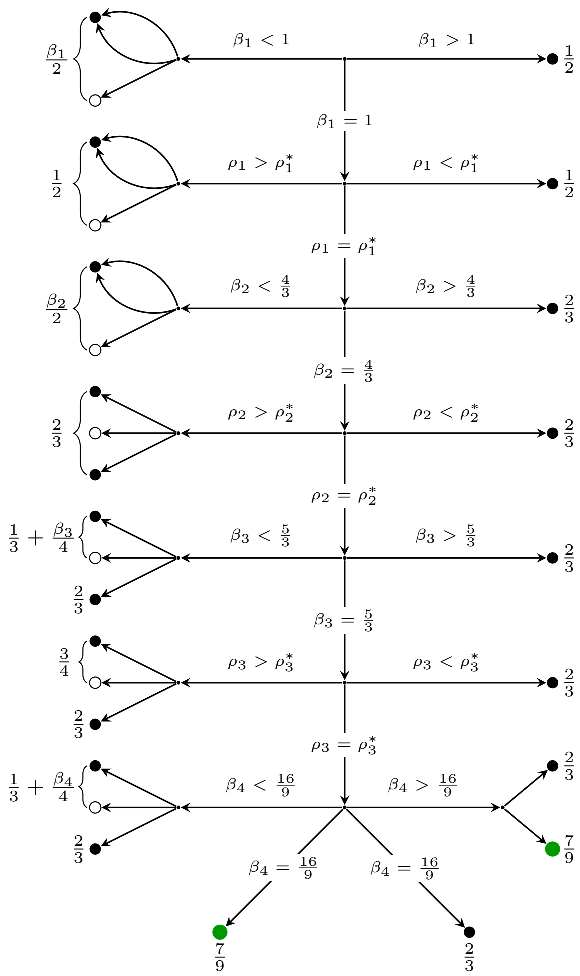
$$\rho_1^* = \frac{3a_3C_0}{|a'_1|}, \quad \rho_2^* = \frac{9}{(12)^{1/3}|a'_1|}(a_3 B^2 C_0^4)^{\frac{1}{3}}, \quad \rho_3^* = 2BC_0\left(\frac{3\rho_2^*}{|a'_1|a_3}\right)^{\frac{1}{2}}. \tag{5.13}$$

We then have the following theorem.

Theorem 5.10. *Suppose V_θ is smooth (in θ) in the space of $\mathcal{C}^8$ functions, and exhibits pitchfork bifurcation at the origin. Suppose also $B \neq 0$. Let u_{ϵ^α} be the solution to the PDE* (5.7) *with initial data $\phi_0^{(\epsilon^\alpha)}$, and let $\theta = \theta(\epsilon)$ be of the form*

$$\theta = \rho_1 \epsilon^{\beta_1} + \rho_2 \epsilon^{\beta_2} + \rho_3 \epsilon^{\beta_3} + \rho_4 \epsilon^{\beta_4}$$

with $0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$ and $\rho_j > 0$. Let ρ_j^*'s be as in (5.13)*. Then, we have the following (with all the limiting processes starting with initial data ϕ_0):*



Here, the notations are the same as in Remark 5.5: each • represents a stable OU process, each ○ represents an unstable OU process, and each green node ● represents a Φ^3_3. Each black node • with two arrows pointing to it indicates that the two limiting OU processes, obtained by shifting the field to the left and to the right, have the same coefficients. The numbers next to each dot indicates the scale α at which one observes the corresponding limit.

Proof. The key in the proof is to show the convergence of $\lambda_j^{(\delta)}$'s as defined in (5.11) to the desired limiting values at various choices of α and h_ϵ. In particular, for the Φ^3_3 limit, the coefficient of the quadratic Wick term is proportional to $B^{\frac{1}{3}}$. The details of the proof are very similar to those in Theorem 5.2, and is straightforward by the expression of the $a_j^{(h)}$'s in (5.8), so we do not repeat them here. $\square$

Remark 5.11. If any of the ρ_j's is negative, it will make θ further away from the effective critical value θ^* (but close to 0), and one could only see one stable OU in the limit. In fact, by including negative ρ_j's, one will fill in the jump of the scale (from $\frac{1}{2}$ to $\frac{2}{3}$) on the right of the figure, and obtain a complete description (in terms of the continuous change of the scale) as the left side of the figure.

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